Toward BEC’s, the Bee’s Knees

* A Computational Investigation
  of Non-ergodic Evaporative Cooling

by

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Figure 1: Evolution of the energy distribution of atoms in a medium-depth Gaussian trap
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Abstract

I present an overview of Bose-Einstein condensation and experimental techniques to cool a gas of atoms to the sub-microkelvin range. We then develop a model of evaporative cooling to describe the evolution of the distribution function of a gas of trapped atoms. Our model extends work in the literature but does not require an assumption of ergodicity. We compute the evolution of a distribution function for atoms in both isotropic Gaussian and harmonic trapping potentials at a variety of trap depths and find that the peak drop in temperature per atom lost is greatest in shallow harmonic traps, but greater in deep Gaussian traps than in deep harmonic traps. Further work towards interpreting our results is suggested.
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Chapter 1

Introduction

This thesis presents an analysis of the dynamics of evaporative cooling toward Bose-Einstein condensation. To provide context and motivation for this work, I present here a brief historical background and mathematical derivation of Bose-Einstein condensation.

1.1 Overview of Bose-Einstein Condensation

In 1925, Satyendra Nath Bose sent a paper on the quantum mechanical statistics governing photons to Albert Einstein, who subsequently translated it into German and had it published in Zeitschrift für Physik. Einstein then went on to apply similar arguments for the statistics governing systems of matter particles, concluding that at sufficiently low temperatures, a system of non-interacting bosons (integer-spin particles) will undergo a phase transition, or “condensation,” so that a large portion of atoms will occupy the lowest-energy state. Such a collection of lowest-energy massive particles is now known as a Bose-Einstein condensate, or BEC [13].

The most obvious impediment to experimentally realizing pure Bose-Einstein condensation is that most substances will condense to a liquid or solid at temperatures much higher than are required to form a BEC. In the liquid and solid phases, interatomic interactions are strong enough to prevent condensation into a BEC altogether, or at least require a significantly modified treatment of the phenomenon. The transition to superfluidity in helium-4, which occurs at a temperature of 2.2 K under atmospheric pressure, can be regarded as an example of Bose-Einstein condensation. However, because this occurs in a liquid state of helium where interatomic interactions are non-negligible, Einstein’s original analysis is not applicable. Similarly, superconductivity and superfluidity in helium-3 can be considered a kind of Bose-Einstein condensation. The particles in
each of these systems (electrons and helium-3 atoms, respectively) are fermions, so they
must pair with each other in order to form a boson. It is these pairs of particles that
condense into a BEC. Superfluidity in helium-3 wasn’t discovered until 1972 because the
pairing mechanism requires a temperature of less than 3 mK before the phase transition
occurs [16]. On the other hand, superfluidity of helium-4 was discovered in 1937 and
superconductivity as early as 1911.

The transition to a liquid or solid can be avoided at low temperatures in a sufficiently
dilute gas. In 1995, a group at JILA led by Eric Cornell and Carl Wieman created the
first BEC out of a gas of weakly interacting rubidium atoms [1]. This work was shortly
followed by Randall Hulet’s group at Rice University, which used attracting lithium
atoms [3], and Wolfgang Ketterle’s group at MIT, which used a gas of sodium atoms [6].
Cornell, Wieman, and Ketterle received the Nobel Prize in 2001 for this work.

In each of these experiments, there are two primary cooling phases: (1) laser cooling
and (2) evaporative cooling. This thesis primarily focuses on the dynamics and optimi-
ization of evaporative cooling. A brief introduction to laser cooling and two different
approaches to evaporative cooling are discussed in the next chapter.

1.2 Derivation of Bose-Einstein Condensation

The following treatment of Bose-Einstein condensation is adapted from Schroeder [18]
and Townsend [20].

Bosons are particles with integer spin. We consider only one type of boson so that
the particles are indistinguishable from one another. Further, the spin-statistics theorem
implies that arbitrarily many bosons may occupy the same quantum state. We would
like to calculate the number of particles \( n(E) \) occupying a particular state with energy
\( E \). It can be shown that indistinguishable particles that are free to occupy the same
state obey Bose-Einstein statistics so that \( n(E) \) is given by

\[
n(E) = \frac{1}{e^{(E-\mu)/kT} - 1}, \tag{1.1}
\]

where \( \mu \) is the chemical potential, \( k \) is Boltzmann’s constant, and \( T \) is the tempera-
ture of the system of particles. This function is shown in Fig. 1.1. To conserve the total
number of particles \( N \), we require

\[
\sum_i n(E_i) = N. \tag{1.2}
\]
In the ground state energy with $E_0 = 0$, we see that

$$N_0 = \frac{1}{e^{(E_0 - \mu)/kT} - 1} = \frac{1}{e^{-\mu/kT} - 1}.$$ 

Rearranging, we find $e^{-\mu/kT} = 1 + 1/N_0$, so that

$$-\frac{\mu}{kT} = \ln\left(\frac{1}{N_0} + 1\right).$$

For $N_0 \gg 1$, to first order in $1/N_0$ we see that $\ln(1 + 1/N_0) \approx 1/N_0$. Finally, this implies $\mu \approx -kT/N_0$. In the large-$N_0$ limit, $-\mu/kT \approx 1/N_0$ is negligible compared to $E/kT$ for all states except the ground state where $E = 0$. In a moment, we will see that the occupation of the ground state is not calculated from the Bose-Einstein distribution Eq. 1.1, so we may freely approximate $\mu \approx 0$ for the remainder of this section.

We define the density of states $g(E)$ to be the number of states per unit energy. Assuming small separations in energy, the total number of particles $N_{E_{\text{min}}}^{E_{\text{max}}}$ between energies $E_{\text{min}}$ and $E_{\text{max}}$ is

$$N_{E_{\text{min}}}^{E_{\text{max}}} = \int_{E_{\text{min}}}^{E_{\text{max}}} n(E)g(E)dE. \quad (1.3)$$

Since the density of states $g(E)$ is generally equal to 0 at $E = 0$, the integral in Eq. 1.3 will not count any states with energy $E \approx 0$ even if there is a significant number of states at low energies. Therefore, if we want to find the total number of particles $N$, we must consider the number of atoms $N_0$ in the ground state (or perhaps several of the
lowest-energy states) separately from the integral:

\[ N = N_0 + \int_0^\infty n(E)g(E)\,dE. \quad (1.4) \]

For simplicity, we will consider a system of particles in a box. It is a standard result that the density of states of a system of particles in a cubic box of side length \( L \) is

\[ g(E) = \frac{(2s + 1)V(2m)^{3/2}}{4\hbar^3\pi^2} \sqrt{E}, \quad (1.5) \]

where \( s \) is the spin of the particles, \( m \) is their mass, \( \hbar \) is the reduced Planck constant, and \( V \equiv L^3 \) is the volume of the box.

Plugging Eq. 1.5 into Eq. 1.4, we see that

\[ N = N_0 + \int_0^\infty \frac{(2s + 1)V(2m)^{3/2}}{4\hbar^3\pi^2} \sqrt{E} \, dE. \]

Substituting \( x \equiv E/kT \),

\[ N = N_0 + \frac{(2s + 1)V(2mkt)^{3/2}}{4\hbar^3\pi^2} \int_0^\infty \frac{\sqrt{x}}{e^x - 1} \, dx. \]

The integral evaluates to approximately \( 1.3\sqrt{\pi} \), so that

\[ N_0 = N - \frac{(2s + 1)V(2mkt)^{3/2}}{4\hbar^3\pi^2} \cdot 1.3\sqrt{\pi}. \quad (1.6) \]

There is one value of \( T \) at which the right-hand side of Eq. 1.6 is equal to 0, which we define to be the critical temperature \( T_C \):

\[ T_C \equiv \frac{2\pi h^2}{mk} \left( \frac{N}{2.6(2s + 1)V} \right)^{2/3}. \quad (1.7) \]

We can rewrite Eq. 1.6 more simply in terms of this quantity as follows:

\[ \frac{N_0}{N} = 1 - \left( \frac{T}{T_C} \right)^{3/2}. \quad (1.8) \]

This formula predicts that there is a significant fraction of atoms in the ground state at a temperature \( 0 \leq T < T_C \). When this happens, we say that Bose-Einstein condensation has occurred. Notice that the number of ground-state atoms is implicitly
monotonically increasing with number density through $T_C$. The fraction of ground-state atoms as a function of temperature is shown in Fig. 1.2

The expression in Eq. 1.8 is slightly misleading since the number density is buried in $T_C$. I will now introduce a more convenient way to characterize a BEC, which will be useful to us later on. First notice that Eq. 1.6 can be rewritten as

$$N_0 = N - 2.6(2s + 1)\left(\frac{\sqrt{2\pi mkT}}{h}\right)^3.$$ 

We identify the term in parentheses as the inverse of the de Broglie wavelength $\lambda_{DB} \equiv h/\sqrt{2\pi mkT}$. The de Broglie wavelength is of the same order of magnitude as the length scale that a particle’s wave function occupies. In terms of $\lambda_{DB}$, Eq. 1.6 becomes

$$\frac{N_0}{N} = 1 - \frac{2.6(2s + 1)}{(N/V)\lambda_{DB}^3}.$$ 

This suggests the definition of phase-space density $\rho_{PS} = (N/V)\lambda_{DB}^3$, which is the denominator of the second term above. Notice that at the critical temperature, $N_0 = 0$ so we have $\rho_{PS} = 2.6(2s + 1)$. We may increase $\rho_{PS}$ by either increasing $N/V$ or decreasing $T$. If $N/V$ increases then more particles will occupy the ground state because $T_C$ increases. Likewise, it is clear from Eq. 1.8 that if $T$ decreases while $N/V$ is held constant, the number of particles occupying the ground state will increase. Thus we see that the phase-space density is a single parameter sufficient to determine whether Bose-Einstein condensation occurs. In particular, a finite fraction of particles occupy

![Figure 1.2: Fractional occupation of the ground state $N/N_0$ as a function of reduced temperature $T/T_C$.](image)
Figure 1.3: When the wavefunctions of atoms do not significantly overlap, we have an ordinary gas (left). When the wavefunctions begin to overlap with each other, Bose-Einstein condensation occurs (right).

Figure 1.4: The possible system states for two indistinguishable bosons (top) and two classical particles (bottom). The black bars represent energy levels, while the circles represent particles.

the ground state exactly when

$$\rho_{PS} > (2s + 1)2.6.$$  \hspace{1cm} (1.9)

This result suggests a useful and intuitive picture. Notice that the right hand side of Eq. 1.9 is a number on the order of unity, while the left-hand side is the number of particles per unit volume times the characteristic volume occupied by each particle’s wave function. In other words, Bose-Einstein condensation occurs when the wavefunctions of particles begin to significantly overlap. This idea is illustrated in Fig. 1.3.

It is interesting to note that this phase transition is qualitatively quite different from the behavior of a system of classical, distinguishable particles. Although at $T = 0$ all of the atoms in a classical gas must occupy the ground state, the proportion of atoms in the
ground state does not become significant until $T$ is almost exactly zero. This is because there is a greater statistical likelihood of distinguishable particles occupying different states from each other. Fig. 1.4 illustrates this statistical pressure for an extremely simple system with two energy levels and two particles. There are two distinct system states in which one particle occupies each energy level in the classical case, whereas these correspond to the same system state if the particles are indistinguishable. Therefore, bosons have a probability equal to $2/3$ of being in the same energy state as each other, while classical particles have a probability equal only to $1/2$ of being in the same state. Loosely speaking, bosons tend to clump into the same energy state more than classical particles do.
Chapter 2

Cooling methods

As mentioned in the Introduction, Bose-Einstein condensation experiments typically involve two stages of cooling: (1) laser cooling and (2) evaporative cooling. The usual procedure is to cool a gas of atoms in what is known as a magneto-optical trap via laser cooling. The atoms are then loaded into a separate trap from which they are evaporatively cooled. This second trap may be magnetic or optical, and both schemes are discussed below. The primary focus of this thesis is the dynamics of a gas of atoms under evaporative cooling, and so laser cooling is given only a cursory overview here.

2.1 Magneto-optical trap

2.1.1 Laser cooling

Steven Chu, Claude Cohen-Tannoudji, and William Phillips received the 1997 Nobel Prize for their development of laser trapping and cooling of atoms. Laser cooling is also known as “Doppler cooling” for reasons that will be apparent shortly [12].

Laser cooling works as follows: laser light is shone on a cloud of atoms from all six directions (left-right, top-bottom, front-back) at a slightly lower frequency than the frequency corresponding to one of the atomic energy transitions. Therefore, the atoms will typically not absorb the light if they are at rest. However, if they are moving toward the source of the light, then the light is Doppler shifted to a frequency closer to their transition, increasing the probability of absorbing a photon (see Fig. 2.1). If they are moving away from the light, then it is red-shifted, further decreasing the probability they will absorb a photon. Therefore, the atoms are most likely to absorb a photon while moving toward the laser light.

When an atom absorbs a photon, it picks up the photon’s momentum. Since the
atom usually absorbs light while opposing the direction of the light, it typically receives a kick of momentum in the direction opposite its original motion. It subsequently re-emits the photon in a random direction. After many absorption-emission events, the momentum kicks from each emission will tend to cancel each other out since they are each in a random direction, while the momentum kicks from each absorption tend to be opposed to the atom’s motion. These kicks serve to slow the atom down no matter which direction the atom is moving. The atoms lose their translational motion; in other words, they become cold.

2.1.2 Zeeman trapping

The cloud of atoms must also be spatially confined. We accomplish this by exploiting what is known as Zeeman splitting. This is a phenomenon in which the energy levels of an atom split according to the projection of the atom’s angular momentum along the axis of a magnetic field. In particular, suppose an external magnetic field points in the $+z$-direction. Then atoms whose angular momentum projection along the $z$-axis is also in the $+z$-direction have lower energy than in the absence of the field; these atoms are said to be in the $m = 1$ state. On the other hand, atoms whose angular momentum
projection along the $z$-axis in the $-z$-direction have higher energy than in the absence of the field; these atoms are said to be in the $m = -1$ state. The atoms whose angular momentum projection along the $z$-axis is 0 have the same energy as if the magnetic field were absent; these atoms are in the $m = 0$ state. This is shown in Fig. 2.2. The magnitude of the energy shift is typically proportional to the strength of the applied field, although nonlinear effects can sometimes complicate the relationship between the energy shift and applied field. Zeeman splitting is discussed in many quantum mechanics textbooks; for an approachable introduction, see [20].

Suppose we create an isotropic linear magnetic field in the region of the atoms so that the magnetic field strength increases away from some central point and reverses direction as that point is crossed. Without loss of generality, let $+z$ be the direction of the magnetic field at a point to the left of the trap’s center. The energy levels of an atom in this region will exhibit Zeeman splitting. Such an atom has lower energy in the $m = 1$ state than in the $m = -1$ state.

In going to an excited state the atom picks up the angular momentum of a photon. If this angular momentum is in the $+z$-direction, then the energy of the excited state is reduced by Zeeman splitting so that the transition frequency is lower than in the absence of a magnetic field.

Now recall that the laser light used for cooling the atoms has a slightly lower energy than one of the natural transition energies of the atoms. By reducing the transition frequency via Zeeman splitting, we can bring the transition frequency and laser frequency closer together, making the absorption of a photon more probable. In particular, we can polarize the photons coming from the left to have angular momentum in the $+z$-direction so that the atoms on the left side of the trap are more likely to absorb these photons.
Figure 2.3: Atoms on the left tend not to absorb light with angular momentum in the $-z$-direction, while they preferentially absorb light with angular momentum in the $+z$-direction since the transition energy to a state with angular momentum along the $z$-axis is near the energy of a photon. If we polarize the right-moving light to have angular momentum in the $+z$-direction then the atoms on the left will preferentially absorb the rightward momentum of these photons and move toward the center of the trap. A similar argument applies to atoms on the right side of the trap.

(see Fig. 2.3). When the atoms on the left side of the trap absorb these right-moving photons, they get a momentum kick back toward the center of the trap.

Since the magnetic field reverses direction as we cross the center of the trap, the energy of an excited atom in the $m = 1$ state on the right side of the trap is increased. Thus the energy required to transition to an excited state with angular momentum in the $+z$-direction is increased even farther above the frequency of our laser for atoms in this region. Therefore, atoms on the right side of the trap will tend not to absorb the right-moving light that we polarized to have angular momentum in the $+z$-direction. As a result, these atoms won’t get preferentially pushed to the right like the atoms from the left side of the trap.

Notice that if we polarize the light coming from the right side of the trap to have angular momentum in the $-z$-direction, there will be a restoring force on atoms from the right side of the trap toward the center. Just as above, there will be no corresponding force on the atoms on the left side of the trap to go farther away from the center. Since the designation of “left” and “right” is completely arbitrary, an identical argument applies to the other pairs of directions, top-bottom and front-back. Thus our magnetic field, paired with properly polarized red-detuned laser light, creates a position-dependent force toward the center of the trap from all directions.
Figure 2.4: Arrangement of MOT lasers and magnets (reproduced from Arolyn Conwill’s undergraduate thesis) The polarizations $\sigma^+$ and $\sigma^-$ correspond to angular momenta aligned and anti-aligned, respectively, with the coordinate axes $+x$, $+y$, or $+z$. It is important not to confuse these designations with right- and left-hand circular polarization states, which are defined by the alignment of the angular momentum along the axis of propagation. In particular, angular momentum is aligned with the axis of propagation in left-hand circularly polarized light and anti-aligned with the axis of propagation in right-hand circularly polarized light. The repump beam shown is necessary to ensure that the atoms remain in a state that may be excited by the main laser beams, but is not crucial to the basic conceptual mechanism of the MOT.

This configuration of magnets and lasers is called a magneto-optical trap (MOT) and is shown in Fig. 2.4. I have provided only a qualitative description of the essential features of a MOT since its behavior is not central to this thesis. For a more thorough analysis at an appropriate level, I urge the reader to see Leon Webster’s undergraduate thesis [21]. There exist further refinements to the process of making a MOT in order to increase the density of trapped atoms. The resulting trap is known as a compression MOT, or CMOT. Such refinements are beyond the scope of this thesis; however, Eric Dodds’s undergraduate thesis describes the CMOT in great detail. [9]

2.2 Evaporative cooling

A gas of particles subject to an external potential energy well has a statistical distribution of energies. The potential energy well is also commonly called a “trap.” If the potential well has a finite depth of energy, then some of the higher-energy atoms will have sufficient energy to permanently escape from the potential well. Since this loss mechanism only selects the high-energy atoms, the remaining atoms will have a lower average energy. The gas then rethermalizes to a lower temperature and the process continues. A diagram illustrating this effect is shown in Fig. 2.5.

The technique of evaporative cooling involves a trade-off between the number of
Figure 2.5: For each plot, the y-axis is energy and the x-axis is distance from the center of the potential energy well. The trapping potential is the bowl-like curve and atoms are represented by black dots. Initially, there are many atoms, some of which have very high energies (left). After a period of evaporation, fewer atoms remain in the trap but their average energy is lower (right).

atoms lost and a drop in temperature. Atoms are also lost from other causes. Most notably, particles at room temperature from the walls of the container will sometimes collide with particles in the trap, giving them an enormous boost in energy that usually removes the atoms from the trap and might also increase the temperature of the gas. Three-body collisions also complicate the analysis of evaporative cooling. These effects are problematic because evaporation is extremely slow once a quasi-equilibrium has been reached. For this reason, it is typical in experiments to incrementally (or even continuously) decrease the depth of the trapping potential so that more atoms are able to escape from the trap in a practical time interval.

Cornell and Wieman’s group used a magnetic trapping potential that superimposes a magnetic field rotating at about 7.5 kHz on top of a static quadrupole field [1]. Since this time scale is much faster than the response time of the atoms, they react almost as if there were a constant trapping potential equal to the time-average of the time-dependent potential. In the JILA BEC experiment, the time-averaged trapping potential was a harmonic potential energy well.

Our group, however, uses an all-optical trap [2]. A CO₂ laser generates a strong electric field at a frequency much lower than any transition frequencies of the rubidium atoms. Since the laser’s electromagnetic field oscillates very slowly compared to the electron oscillations in the atoms, the laser merely has the effect of a strong external electric field. Due to polarization effects, the atoms are drawn in toward the focus of the laser beam where the electric field is strongest. The shape of the trapping potential is the same as the intensity profile of the laser, which is radially Gaussian and axially Lorentzian. We can adjust the depth of the potential well by changing the power of the
laser. The maximum power output of the laser is about 100 W, although we do not typically use more than about 40 W to trap atoms.

Luiten, _et al._ apply kinetic theory to a dilute gas under the influence of a trapping potential energy well [14]. One of the purposes of this thesis is to build on their work by computing the evaporative cooling trajectory for different trapping potential shapes (in particular, harmonic and Gaussian) and depths (small and large maximum potential energies compared to $kT$) based on iteratively applying the kinetic equation that governs the distribution of atomic energies. In order to prepare for this task, I describe the operation of both harmonic magnetic and Gaussian optical dipole traps in more mathematical detail below.

### 2.3 Optical dipole trap

The optical dipole trap induces a dipole moment in trapped atoms via a strong external electric field and the atoms are then attracted to the point of strongest electric field. Specifically, our strong electric field is a high-power laser beam. Although a quantum mechanical calculation is appropriate, this technique is essentially classical. In particular, the results of a classical calculation are the same as for a quantum mechanical second-order perturbation calculation [10]. I therefore present the more intuitive classical picture.

Model a trapped atom as a damped oscillator driven by the external electric field of a laser beam, where the atom’s natural resonance occurs at the frequency corresponding to an electronic transition. Although this model breaks down for an extremely low driving frequency relative to the atom’s resonant frequency, which is the regime most applicable to our lab, the essential mechanism of polarizing the atoms and drawing them into a strong electric field is clearly conveyed by the following treatment.

It is a standard result of mechanics that a damped harmonic oscillator will oscillate in phase with the driving frequency as long as the driving frequency is far below the oscillator’s resonant frequency; for example, see the discussion in [19]. Induced electric dipoles have lower energies when aligned with an electric field than in the field’s absence according to the potential energy equation

$$U_{dip} = -\frac{1}{2} \langle \vec{p} \cdot \vec{E} \rangle,$$  

(2.1)

where $\vec{p}$ is the induced dipole moment, $\vec{E}$ is the electric field, and the brackets indicate a time-average. Since the trapped atoms oscillate in phase with the external field,
they are subject to a deeper potential well as the electric field strength increases. In particular, it can be shown that in the regime where the driving frequency is far below the resonant frequency of the atom, the shift towards lower energy $\Delta E$ obeys the following relationship [10]:

$$\Delta E \propto -\frac{I}{\delta},$$

(2.2)

where $I$ is the intensity of the laser beam and $\delta = \nu_{\text{res}} - \nu_{\text{drive}}$ is the detuning of the driving frequency from the resonant frequency. The intensity gradient of the laser beam results in a potential gradient for the atoms; i.e., a force. Since the intensity is strongest at the center of the laser beam and falls off radially, the intensity gradient results in an inward central force on the atoms. Thus the atoms are trapped by the laser beam.

We may not increase this force by reducing the detuning $\delta$ without repercussions because the scattering rate $\Gamma_{\text{sc}}$ of laser light by the atoms obeys the relation [10]

$$\Gamma_{\text{sc}} \propto \frac{I}{\delta^2}.$$  

(2.3)

When atoms scatter light, they are usually lost from the trap, so we want to minimize $\Gamma_{\text{sc}}$. The solution is to make $\delta$ very large so that the scattering rate is negligible and to compensate in Eq. 2.2 by using a very intense laser beam.

In our lab, we use a 100 W CO$_2$ laser with wavelength 10.6 $\mu$m. This is sufficiently detuned from any resonant transitions in Rb that scattering events occur at most once every five minutes. We typically only use 20-50 W of power in the trap.

The exact force on an atom depends on the intensity profile of the laser beam. We use ordinary Gaussian optics, for which the intensity of a focused beam propagating along the $z$-axis is given by

$$I(r) = I(z) \exp\left(-\frac{r^2 I_0}{w_0^2 I(z)}\right),$$

(2.4)

where $r$ is the distance from the axis of the beam, $I_0$ is the peak intensity, and

$$I(z) = \frac{I_0}{1 + \left(z/z_r\right)^2},$$

(2.5)

Here $z_r$, known as the Rayleigh range, is the distance from the axis of the beam at which the intensity is half the peak intensity.
Figure 2.6: Left to right: the potential energies felt by atoms in $m = -1, 0, \text{ and } 1$ states, respectively, when a magnetic field is applied. The horizontal axis is position and the vertical axis is potential energy. Only atoms in the $m = -1$ state are trapped by this set of state-dependent potentials.

2.4 Magnetic trap

Magnetic traps are essentially a simplified version of the Zeeman trapping discussed above. If a magnetic field is applied in the $z$-direction in a region containing a cloud of atoms with non-zero angular momentum, then the atoms experience Zeeman splitting with energies distinguished according to the projection of their angular momenta along the $z$-axis. Now suppose that a spatially-varying magnetic field is created with a global minimum at some point that we refer to as the center of the trap. The resulting potential energy for atoms in each angular momentum state are shown in Fig. 2.6. Atoms in the $m = 0$ state are unaffected by this potential. On the other hand, the atoms in the $m = 1$ state have lower energy where the magnetic field is large. These atoms will experience a force away from the trap’s center, which is at the point of minimum magnetic field and thus (for them) maximum potential energy. On the other hand, atoms in the $m = -1$ state have higher energies where the magnetic field is large. Therefore, these atoms experience a force toward the trap’s center, which is at (for them) the point of minimum potential energy. This method has the obvious disadvantage of “throwing out” the atoms in the $m = 0$ and $m = 1$ states, but has nevertheless proved successful in many experiments.

Many different magnet configurations may be used to produce a variety of trapping potentials. The one we shall consider is the time-averaged, orbitting potential (TOP) trap. There are several reasons for this choice: (1) the potential has a similar position-dependence in the radial and axial directions, making it suitable for the approximation of isotropicity that we will need later; (2) further, the TOP trap is the only magnetic trap that may be modified to create a spherically symmetric trapping potential; (3) the
first $^{87}$Rb BEC at JILA was produced in a TOP trap and this trapping configuration is highly relevant in the field; and (4) the trapping potential is harmonic so that many other traps have a locally similar potential energy near their center. In particular, the radially Gaussian potential of an optical dipole trap is locally harmonic near the beam axis.

The TOP trap is a modification of the spherical quadrupole trap, which involves a simple arrangement of magnets producing a potential linear in each cartesian coordinate. The primary difficulty with the spherical quadrupole trap is that atoms can experience what is known as a nonadiabatic spin flip in the region where the magnetic field is zero. When this occurs, the atoms transition to an untrapped state and are lost. Adding a constant bias field merely offsets the potential energy and does not change the dynamics of the system. One solution to this is to superimpose a time-dependent magnetic field that moves the location of field’s zero point too quickly for the atoms to respond. By superimposing the field described in Petrich, et al. onto the field of a spherical quadrupole trap and time-averaging the resulting potential energy, the following relationship is obtained [17]:

$$U_{TOP} \propto r^2 + 8z^2 \quad (2.6)$$

where $r$ is the familiar radial coordinate, $z$ is the axial coordinate of the trap, and $\propto$ indicates “approximately proportional to.”

Evaporative cooling may be carried out in this trap by inducing an energy-dependent transition to an untrapped state. This is commonly achieved by applying a radiofrequency field that induces spin-flip transitions in atoms [7]. Although this field is applied throughout the cloud of atoms, the resonance for inducing a transition depends on the magnetic field so that atoms may be removed at a specific equipotential in the magnetic trap. Essentially all atoms that reach this equipotential are removed, which effectively truncates the potential energy in Eq. 2.6 at some spatial boundary. This effect is known as the “RF knife.” The remaining atoms rethermalize and the resonance point may be reduced to a smaller equipotential to force further evaporative cooling.
Chapter 3

Evaporation kinetics

I now begin a detailed discussion of the behavior of a gas of atoms subject to evaporative cooling. We expand on techniques commonly used in the literature by modifying a kinetic equation governing the evolution of a trapped gas.

3.1 Preliminaries

My goal for remaining chapters is to computationally study the evolution of a gas trapped in a dipole-like or TOP-like trap. I will then compare evaporation in each type of trap and discuss the effects of trap depth on the cooling trajectory.

We prepare for this task by first introducing a kinetic equation that governs the evolution of a gas of particles. Following Luiten, et al., we consider a gas of $N$ particles of mass $m$ subject to a potential well $U(r)$ undergoing random thermal collisions [14].

3.1.1 Assumptions

We assume that the gas is in the Knudsen regime so that the size of the gas cloud is very small compared to the collisional mean free path of the atoms. This implies that any particle in a state that may escape the trap does so with nearly unit probability.

Although the ultimate goal of many evaporative cooling experiments is to obtain a phase-space density on the order of unity where quantum effects dominate (see the Introduction), the temperature of a gas must be reduced by many orders of magnitude while still in the classical regime before reaching this point. Therefore, we assume the phase-space density is small compared to unity and classical statistics apply. As a result, our model should be expected to break down very near the regime of Bose-Einstein condensation.
Further, we must model the scattering of particles as they interact with each other. In general, this is a very complicated process. However, consider a low-energy atom scattering to some outgoing wave off of a target particle. At the low temperatures characteristic of evaporative cooling, the de Broglie wavelength of the atom is too large to resolve any structure in the target and only long-range interactions affect the scattering process. At large distances, any interaction potential appears roughly spherical and can be characterized by a single quantity known as the scattering length. More formally, the outgoing scattered wave can be expanded analogously to the multipole expansion of a potential in classical electrodynamics, and at low temperatures the component with zero angular momentum dominates. Scattering dominated by the zero-angular momentum term in this expansion is known as $s$-wave scattering. If the target in an $s$-wave scattering event is a hard sphere of radius $R$, then the scattering length is $a = R$. Thus a target with scattering length $a$ has the same scattering properties at low energies as a hard sphere of radius $a$.

Evaporative cooling is carried out at sufficiently low temperatures to justify the assumption of pure $s$-wave scattering between atoms. Thus in computing the scattering cross section, we can neglect contributions to the outgoing scattered wave from terms with angular momentum greater than $l = 0$ to find that the scattering cross section is $8\pi a^2$. Notice that this cross section is independent of energy, which simplifies the analysis considerably.

It is common in the literature to assume a high degree of ergodicity, meaning the distribution of atoms in phase-space depends only on their energies [11, 14, 15]. In this model, an atom with energy $\epsilon > \epsilon_t \equiv U_{\text{max}}$ will sample an untrapped state with unit probability, allowing for the truncation of the energy distribution of atoms at the trap depth $\epsilon_t$. We will have much more to say about ergodicity later, but for the discussion immediately following we assume complete ergodicity in our system.

### 3.1.2 Introduction to the distribution function

We are not currently interested in the trajectories of atoms through the space of the trap. Rather, we are concerned with statistical distribution of the atoms’ energies. We therefore define the distribution function, familiar from elementary statistical mechanics, to be the number of particles with energy between $\epsilon$ and $\epsilon + d\epsilon$. In particular, we can write such a distribution function as $\rho(\epsilon)f(\epsilon)$, where $\rho(\epsilon)$ is the number of states with energy between $\epsilon$ and $\epsilon + d\epsilon$ and $f(\epsilon)$ is the probability that an atom is in some particular state that has an energy between $\epsilon$ and $\epsilon + d\epsilon$. Another way to say this is that $f(\epsilon)$ is
the probability that an atom would have energy $\epsilon$ if there were only one state to occupy in each infinitesimal energy range $\epsilon + d\epsilon$; with this view, $\rho(\epsilon)$ corrects for the fact that an atom is more likely to have a particular energy if there are more states corresponding to that energy. When $N \gg 1$ and we can neglect the statistical effects arising from the indistinguishability of similar atoms in quantum mechanical systems, $f(\epsilon)$ is known as the Maxwell-Boltzmann distribution and is given by

$$f(\epsilon) = ne^{-\epsilon/kT},$$

(3.1)

where $n$ is a normalization constant that conserves number (see Fig. 3.1). This function may be derived by applying simple counting arguments to the distribution of energy among $N$ distinguishable particles and taking the limit as $N \to \infty$.

### 3.2 The kinetic equation

Luiten, et al. model the collision between particles with energies $\epsilon_3$ and $\epsilon_4$ resulting in particles with energies $\epsilon_1$ and $\epsilon_2$. Considering the evolution of the phase-space (i.e., position- and momentum-space) distribution of a classical gas, and assuming ergodicity, they arrive at the following kinetic equation for the time evolution of a distribution function:

$$\rho(\epsilon_4) \dot{f}_{\text{kin.}}(\epsilon_4) = 64\pi ma^2h^{-3} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \times \rho(\min[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]) \{f(\epsilon_1)f(\epsilon_2) - f(\epsilon_3)f(\epsilon_4)\},$$

(3.2)
where $h$ is Planck’s constant. The Dirac delta function ensures that energy is conserved, \( \rho(\min[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]) \) limits collisions to regions that are energetically accessible, and \( f(\epsilon_1)f(\epsilon_2) \) is related to the probability of particles obtaining energy \( \epsilon_4 \) through thermal collisions, whereas \( f(\epsilon_3)f(\epsilon_4) \) is related to the probability of knocking atoms with energy \( \epsilon_4 \) into a different energy state during collisions.

In principle, all we need to do now is pick a trapping potential, find the corresponding density of states, and numerically integrate this equation to find the time evolution of the energy distribution of a trapped gas. Toward this goal, we now consider the density of states function.

The total density of states is

\[
\rho(\epsilon) \equiv \frac{1}{h^3} \int d^3r d^3p \delta(\epsilon - U(r) - p^2/2m).
\] (3.3)

The delta function ensures that we count exactly those states in phase-space that conserve energy. We can evaluate the momentum integral to obtain [14]

\[
\rho_{\text{all}}(\epsilon) = \frac{2\pi(2m)^{3/2}}{h^3} \int_{U(r) \leq \epsilon} d^3r \sqrt{\epsilon - U(r)}.
\] (3.4)

Notice that the integral runs over all positions where the particle can have non-negative energy. Further, it should be clear that all states with \( \epsilon \leq \epsilon_t \) are trapped since a particle with energy in this range cannot have positive kinetic energy at the boundary of the trap where \( U(r) = \epsilon_t \). Some groups, assuming complete ergodicity, truncate the density of states at the trap depth \( \epsilon_t \) and simply use the formula above for their density of states [6,14]. However, it is not necessarily true that any state with \( \epsilon > \epsilon_t \) is untrapped: for example, an atom may stably orbit in a circle at some radius \( r \) such that \( U(r) = \epsilon_t \). Then it will have energy \( \epsilon = U(r) + p^2/2m = \epsilon_t + p^2/2m > \epsilon_t \) but will not leave the trap.

This example shows that whether a particle is trapped depends on both its energy \( \epsilon \) and angular momentum \( L \). Thus to find the density of trapped states in a non-ergodic system, we should only integrate over a limited range of angular momenta in Eq. 3.3. This is to be distinguished from how we found the density of all states in Eq. 3.4, where we integrated over all momentum states.

If we limit the range of angular momenta to count only trapped states in an isotropic trapping potential (meaning the potential energy depends only on the distance \( r \) from.
some central point), then Eq. 3.3 becomes [8]

\[
\rho_{\text{trapped}}(\epsilon) = \frac{4\pi^2 \sqrt{2m}}{\hbar^3} \int_{r_{\text{min}}(\epsilon)}^{r_{\text{max}}(\epsilon)} \int_{L_{\text{min}}^2(r,\epsilon)}^{L_{\text{max}}^2(r,\epsilon)} \frac{d(L^2)dr}{\sqrt{\epsilon - L^2/(2m^2) - U(r)}}. \tag{3.5}
\]

Here \( r_{\text{max}}(\epsilon) \) is the maximum radius a trapped particle can reach with a given energy and \( r_{\text{min}}(\epsilon) \) is the minimum radius a trapped particle can reach with a given energy. Similarly, \( L_{\text{max}}^2(r,\epsilon) \) is the maximum squared angular momentum accessible to a trapped particle with energy \( \epsilon \) at radius \( r \) and \( L_{\text{min}}^2(r,\epsilon) \) is the minimum squared angular momentum accessible to a trapped particle with energy \( \epsilon \) at radius \( r \).

Notice that if we are to use Eq. 3.5, then we must lose some generality by assuming an isotropic trapping potential. Although modelling a potential as isotropic makes the problem of determining when a state is trapped much easier and allows computations to be much faster, most traps (including the TOP and dipole traps discussed in Ch. 2) are at least slightly anisotropic.

Since the kinetic equation Eq. 3.2 was derived assuming ergodicity, we must modify it to be consistent with our use of Eq. 3.5, where we allow for trapped particles that have sufficient energy to escape the trap. In particular, if a particle is promoted to energy \( \epsilon_4 \) through thermal collisions, we will only count it as contributing to the distribution function of our trapped gas if it is in one of the states at energy \( \epsilon_4 \) that remain in the trap. Since the term \( f(\epsilon_1)f(\epsilon_2) \) is proportional to the number of particles that get promoted to energy \( \epsilon_4 \), we need to multiply the contribution of this term to the distribution function by a factor of \( \rho_{\text{trapped}}(\epsilon_4)/\rho_{\text{all}}(\epsilon_4) \). In other words, the ratio of atoms promoted to energy \( \epsilon \) that we count as in the trap is the same as the ratio of trapped states to total states at energy \( \epsilon \). The modified kinetic equation is reproduced here:

\[
\rho(\epsilon_4)f_{\text{kin.}}(\epsilon_4) = 64\pi ma^2\hbar^{-3} \int d\epsilon_1d\epsilon_2d\epsilon_3 \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \\
\times \rho(\min[\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4]) \{ \frac{\rho_{\text{trapped}}(\epsilon_4)}{\rho_{\text{all}}(\epsilon_4)} f(\epsilon_1)f(\epsilon_2) - f(\epsilon_3)f(\epsilon_4) \}. \tag{3.6}
\]

3.3 Other considerations for the distribution function

In addition to the changes to the distribution function \( \rho(\epsilon)f(\epsilon) \) due to thermal collisions, we consider the effect of background collisions. Since trapped particles have energies much less than particles at room temperature, the majority of collisions with background atoms in the experimental chamber result in immediate atom loss. The rate of particle

\footnote{There is a discrepancy between this equation and the one appearing in [8]. The paper is missing a factor of 4/\pi, and I have printed the corrected version.}
loss is non-negligible, but small compared to the loss due to evaporation. The background loss rate is independent of an atom’s energy, so we must uniformly subtract a small fraction of the distribution function:

$$\rho(\epsilon)\dot{f}_{bg}(\epsilon) = -L\rho(\epsilon)f(\epsilon), \quad (3.7)$$

where $L$ is a small constant in units of number per time. The precise value of $L$ will vary for every experimental system. We have chosen $L$ such that the loss due to background collisions is on the order of 1% of the loss due to evaporation.

There is also a contribution to $\rho(\epsilon)\dot{f}(\epsilon)$ from ergodic mixing. A trapped particle in an ergodic system will sample an untrapped state at the same energy with probability proportional to $\rho_{\text{untrapped}}(\epsilon)/\rho_{\text{all}}(\epsilon)$ and escape, where $\rho_{\text{untrapped}}(\epsilon) = \rho_{\text{all}}(\epsilon) - \rho_{\text{trapped}}(\epsilon)$. We define the ergodicity parameter $\xi$ such that

$$\rho(\epsilon)\dot{f}_{\text{erg}}(\epsilon) = -\xi\frac{\rho_{\text{untrapped}}(\epsilon)}{\rho_{\text{all}}(\epsilon)}\rho(\epsilon)f(\epsilon), \quad (3.8)$$

subject to the constraint that $\Delta t\xi\frac{\rho_{\text{untrapped}}(\epsilon)}{\rho_{\text{all}}(\epsilon)} \leq 1$ for all $\epsilon$ so that the number of atoms with energy $\epsilon$ lost due to ergodic mixing in a time step $\Delta t$ does not exceed the number of atoms at that energy. Note that for $\epsilon < \epsilon_t$, we have $\rho(\epsilon)\dot{f}_{\text{erg}}(\epsilon) = \rho_{\text{untrapped}}(\epsilon) = 0$. For a perfectly ergodic system, $\xi$ is arbitrarily large so that all atoms in the tail sample an untrapped state and escape in an infinitesimal time interval. Note also that depending on the system dynamics, $\xi$ could be a function of temperature, density, or several other factors. We do not consider these complications here, and instead assume that ergodic effects are negligible. This implies that a trapped state never samples an untrapped state unless it is disturbed through a collision, and there is thus no loss due to ergodic mixing. This is achieved exactly when $\xi = 0$, which is what we use in the following work. The assumption of a non-ergodic system is suggested by the findings of Davis, et al., which are based on a study of an evaporatively cooled gas of sodium atoms [6]. However, I emphasize that one of the strengths of our model is that it can be applied to a system with any degree of ergodicity.

The full expression for the change in the distribution function is then

$$\rho(\epsilon)\dot{f}(\epsilon) = \rho(\epsilon)\dot{f}_{\text{kin}}(\epsilon) + \rho(\epsilon)\dot{f}_{bg}(\epsilon) + \rho(\epsilon)\dot{f}_{\text{erg}}(\epsilon). \quad (3.9)$$
3.4 Computational techniques

Our strategy is to consider a gas that has been loaded into an evaporative cooling trap and then evolve it for a short time step using Eq. 3.9. We then calculate the new temperature and number of remaining atoms for use in the Maxwell-Boltzmann distribution. This recalculation of temperature and number of atoms models rethermalization in the evaporative cooling process. We then evolve the new distribution function for another short time step, and iterate this procedure until the temperature stops significantly falling.

We assume the gas is in thermal equilibrium with quasi-temperature $T_0$ at the beginning of the evaporation process, so that the energies of $N_0$ atoms are distributed according to the distribution function $\rho(\epsilon)f(\epsilon)$. The shape of the distribution function is uniquely determined for a given trapping potential by the quasi-temperature $T$ and normalization constant $n$ in $f = ne^{-\epsilon/kT}$. We discuss how to calculate these parameters immediately below; the density of states $\rho(\epsilon)$ will be calculated in the next chapter for Gaussian and harmonic traps.

During the $i$-th time step,

$$N_i = \int \rho f_i d\epsilon = n_i \int \rho e^{-\epsilon/kT_i} d\epsilon$$ \hspace{1cm} (3.10)

and

$$E_i = \int \epsilon \rho f_i d\epsilon = n_i \int \epsilon e^{-\epsilon/kT_i} d\epsilon.$$ \hspace{1cm} (3.11)

The first equation simply says that the number of atoms $N_i$ in the trap during the $i$-th time step is obtained by summing the number of atoms between energies $\epsilon$ and $\epsilon + d\epsilon$ over all energies $\epsilon$. In the second equation, each atom’s energy $\epsilon$ contributes to the total energy $E_i$ of the system, and again we count the atoms over all energies.

The average energy is then

$$\bar{E}_i(T_i) \equiv \frac{E_i}{N_i} = \frac{\int \epsilon \rho e^{-\epsilon/kT_i} d\epsilon}{\int \rho e^{-\epsilon/kT_i} d\epsilon}.$$ \hspace{1cm} (3.12)

This expression for the average energy depends only on the temperature $T$ and the trapping potential (through the density of states $\rho$), and is typically monotonically increasing in $T$. We can numerically find $\bar{E}(T)$ for a number of different values of $T$ and interpolate these points into a smooth invertible function $\bar{E}(T)$ (see Fig. 3.2). Thus if we can find $\bar{E}_{i+1}$, then we can find the corresponding value of $T_{i+1}$. It will be easiest to do this by
Figure 3.2: Average energy as a function of temperature in Gaussian (left) and harmonic traps (right). For each trap, $E/E_0$ is shown as a function of $T/T_0$ when $\epsilon_t = kT_0$ (top curve) and $\epsilon_t = 7kT_0$ (bottom curve).

To find $E_{i+1}$ and $N_{i+1}$ and then taking their ratio. We know that

$$N_{i+1} \approx N_i + \Delta t \int \rho \dot{f} \, d\epsilon,$$

$$E_{i+1} \approx E_i + \Delta t \int \epsilon \rho \dot{f} \, d\epsilon + \Delta t H N_i.$$

(3.13)

To interpret the first equation, notice that $\rho \dot{f}$ is the rate of change of the distribution function. Then $\Delta t \rho \dot{f}$ is approximately the change in the distribution function in time step $\Delta t$. This identification becomes strictly true as $\Delta t \rightarrow 0$. Therefore, $\Delta t \int \rho \dot{f} \, d\epsilon$ adds up the change in the number of atoms between energy $\epsilon$ and $\epsilon + d\epsilon$ over all energies. We may then identify $\Delta t \int \rho \dot{f} \, d\epsilon$ as the change in the total number of atoms $\Delta N$ between the $i$-th and $(i+1)$-th time steps.

Likewise, $\Delta t \int \epsilon \rho \dot{f} \, d\epsilon$ counts the change in the energy of the system between the $i$-th and $(i+1)$-th time steps due to changes in the distribution function. The term $\Delta t H N$ accounts for systematic heating effects, which are generally proportional to the number of atoms. Thus $H$ may be interpreted as the rate at which energy is systematically deposited into each atom. We have chosen $H$ such that the rate of cooling due to evaporation is balanced by the heating effects for a medium-depth Gaussian trap with $\epsilon_t/kT_0 = 5$ when the temperature drops to $T = 0.1\epsilon_t/k$. This is a reasonable order of magnitude for heating effects since it has been observed that cooling becomes negligible at temperatures around one tenth of the trap depth [15].

Given the density of states $\rho(\epsilon)$ and the Maxwell-Boltzmann distribution at the $i$-th time step $f_i(\epsilon, n_i, T_i)$, we can use Eq. 3.9 to compute $\rho \dot{f}$. We then use Eq. 3.13 to find the
energy $E_{i+1}$ and number of atoms $N_{i+1}$ in the system a short time later, which we can use in Eq. 3.12 to compute the new average energy $\bar{E}_{i+1}$. We can use this together with the relationship between average energy and temperature to find the new temperature $T_{i+1}$. Then, we can compute $n_{i+1}$ from the following equation applied to the $(i + 1)$-th time step:

$$n_i = \frac{N_i}{\int \rho e^{-\epsilon/kT_i} d\epsilon},$$  

(3.14)

which we find by simply rearranging Eq. 3.10. We may then repeat the entire process.

We have chosen to vary the length of each time step so that the total energy drops by 1% of the original energy each step. This has several advantages: the evolution is neither particularly fast in the beginning nor particularly slow in the end, the process obviously cannot continue for more than 100 steps, and the dynamics of the system do not significantly change between one step and the next. There are physical considerations for the time scale of evolution that the kinetic equation cannot accommodate. For example, a shallower trap will have a lower oscillation frequency and thus will thermalize more slowly than an otherwise equivalent system with a higher trap depth [2]. So even though the relative time scale of steps will be consistent for a given calculation, our computational step sizes do not translate reliably into an experimental time scale.
Chapter 4

Density of states

As we can see from Eq. 3.6, all information about the trapping potential relevant to the evolution of the distribution function can be extracted from the density of states. However, to calculate the density of states, we need to find the appropriate limits of integration in equations 3.4 and 3.5. This chapter introduces the trapping potentials we will use to model evaporation in TOP and optical dipole traps. We will then find the limits of integration needed to calculate the density of states.

We consider isotropic Gaussian and harmonic potentials to model dipole and TOP traps, respectively. The Gaussian potential is given by

\[ U(r) = \epsilon_t (1 - e^{-(r/\sigma)^2}) \]  

(4.1)

where \( \sigma \) is the 1/e radius of the trapping potential. The harmonic potential is

\[ U(r) = \epsilon_t (r/\sigma)^2. \]  

(4.2)

The coefficient \( \epsilon_t/\sigma^2 \) of the harmonic potential is chosen so that the Gaussian and harmonic potentials have the same curvature at \( r = 0 \). Therefore, the traps are equivalent in the limit of small displacements from their centers. These trapping potentials are shown in Fig. 4.1.

We consider the limits of integration for each potential separately.
4.1 Limits of integration: the harmonic trap

We will now find the limits of integration in Eq. 3.4 for the harmonic trap. We make a distinction between the energy ranges $\epsilon \in [0, \epsilon_t]$ and $\epsilon > \epsilon_t$. We refer to the region $\epsilon \in [0, \epsilon_t]$ as the head and we refer to the region $\epsilon > \epsilon_t$ as the tail of the density of states. Since all states are trapped states in the head region, we don’t need to bother finding the limits of integration in Eq. 3.5 for $\epsilon \in [0, \epsilon_t]$.

First, notice that for $\epsilon \in [0, \epsilon_t]$, a particle may always be trapped in a state with zero angular momentum so that $r_{\text{min}}(\epsilon) = 0$. Further, $r_{\text{max}}(\epsilon)$ is obtained by the state with zero angular momentum at the turning point in its orbit. For a particle with energy $\epsilon$, the turning point occurs at the radius $r$ satisfying $U(r) = \epsilon = \epsilon_t (r/\sigma)^2$. Then for the head region, we find $r_{\text{max}}(\epsilon) = \sigma \sqrt{\epsilon/\epsilon_t}$. This is sufficient to find $\rho_{\text{trapped}}(\epsilon) = \rho_{\text{all}}(\epsilon)$ in the head region.

Further, recall that in harmonic traps, the “RF knife” causes all atoms to leave the trap when they pass some radius $r_{\text{wall}}$ such that $U(r_{\text{wall}}) = \epsilon_t$. In other words, the depth of the trap is defined by the potential at the point where the RF knife is activated. For the harmonic trap defined above, $r_{\text{wall}} = \sigma$. Therefore the spatial range of the trap is only $r \in [0, \sigma]$. Thus we may use this range for the limits of integration for $\rho_{\text{all}}(\epsilon)$ even when $\epsilon > \epsilon_t$. We show $r_{\text{max}}(\epsilon)$ in Fig. 4.2.

All that remains is to find the limits of integration for trapped states in the region $\epsilon > \epsilon_t$. Consider some orbit of a particle with energy $\epsilon$ that passes through radius $r$. If this particle has a non-zero radial velocity as it passes through $r$, we can increase its
angular kinetic energy at the expense of its radial kinetic energy. This only becomes impossible when the radial kinetic energy is 0; in other words, the particle’s velocity has no radial component. So we see that $L_{\text{max}}$, the maximum angular momentum of a particle passing through radius $r$ with energy $\epsilon$, is given by an orbit with zero radial velocity at radius $r$.

Before solving for $L_{\text{max}}^2$, it is expedient to introduce the effective potential energy

$$U_{\text{eff}}(r, L) = U(r) + \frac{L^2}{2mr^2}.$$  \hspace{1cm} (4.3)

We will sometimes consider $U_{\text{eff}}(r, L)$ to only be a function of $r$ when considering orbits with fixed angular momentum $L$. The effective potential energy allows us to model the three-dimensional motion of a particle subject to potential energy $U(r)$ with fixed angular momentum $L$ as if it were a particle moving in one dimension subject to potential energy $U_{\text{eff}}(r)$. Information about this technique can be found in most mechanics textbooks; for instance, see [19]. The effective potential energy functions are shown for both harmonic and Gaussian traps at a variety of angular momenta in Fig. 4.3.

For the orbit passing through $r$ with energy $\epsilon$ with the maximum angular momentum, we know $\epsilon = U_{\text{eff}}(r) + \frac{mv_r^2}{2} = U_{\text{eff}}(r)$, where $v_r = 0$ is the radial component of the particle’s velocity. Plugging in Eq. 4.2 for $U(r)$, we can solve $\epsilon = U_{\text{eff}}(r, L_{\text{max}})$ to find $L_{\text{max}}^2(r, \epsilon) = 2mr^2(\epsilon - \epsilon_t(r/\sigma)^2)$. This is shown as a function of radius in Fig. 4.4 and as a function of energy in Fig. 4.5.

Unlike in the head region, an atom with $L = 0$ will escape the trap for $\epsilon > \epsilon_t$. As we
increase the angular momentum from $L = 0$, eventually the effective potential energy curve intersects the line of constant energy $\epsilon$ at radius $r = \sigma$ so that the particle has an outer turning point at the trap wall. Then $L_{\min}^2$ is determined by $\epsilon = U_{\text{eff}}(\sigma)$, which gives $L_{\min}^2(r, \epsilon) = 2m\sigma^2(\epsilon - \epsilon_t)$. This is shown in Fig. 4.6.

There is actually a subtle issue to address before we accept this result for $L_{\min}$. In general, a potential energy function might grow slowly enough at large distances that the effective potential energy curve peaks at some finite value and then begins to decrease as the angular momentum term $L^2/(2mr^2)$ goes to zero with increasing $r$. In this situation,
as $L$ is increased from 0 we might find that the local peak in the effective potential energy curve intersects the line of constant energy at some radius $r < \sigma$ inside the trap wall. However, this cannot happen if the curvature of $U_{\text{eff}}(r)$ is positive everywhere. For the harmonic trap, $d^2U_{\text{eff}}(r)/dr^2 = 2\epsilon_t/\sigma^2 + 3L^2/(mr^4) > 0$. Therefore the result $L_{\text{min}}(r, \epsilon) = 2m\sigma^2(\epsilon - \epsilon_t)$ is valid.

From the way we found $L_{\text{min}}$, it is clear that $r_{\text{max}}(\epsilon) = r_{\text{max}} = \sigma$. The inner turning point occurs at the other radius where $\epsilon = U_{\text{eff}}(r, L_{\text{min}})$. Solving this, we find
The minimum trappable radius at energy $\epsilon$ in a harmonic trap is the inner turning point corresponding to the orbit with the lowest trappable angular momentum at that energy

$$r_{\text{min}}(\epsilon) = \sigma \sqrt{(\epsilon - \epsilon_t)/\epsilon_t}.$$  This is shown in Fig. 4.7.

The last issue I will address before calculating the density of states in a harmonic trap is the maximum energy possible for a trapped state. Any trapped particle with $\epsilon > \epsilon_t$ must have inner and outer turning points somewhere in the trap. Notice that if these turning points are not equal (i.e., the orbit is not a circle) then we may increase the angular momentum while holding the energy constant to bring the turning points closer together. Eventually, the turning points meet and we have a circular orbit. Therefore any highest-energy orbit will either be a circular orbit or be at the same energy as some circular orbit. Thus we can consider all circular orbits at radius $r \in [0, \sigma]$ and choose the one with the highest energy.

A circular orbit with radius $r$ is defined by the fact that a particle has no radial velocity, so $\epsilon = U_{\text{eff}}(r)$, together with the fact that there is no net radial force on the particle, so $dU_{\text{eff}}(r)/dr = 0$. This second condition gives $2\epsilon_tr/\sigma^2 - L^2/(mr^3) = 0$, which we can use to solve for $L_{\text{circle}}^2(r) = 2m\epsilon_t r^4/\sigma^2$. If we plug this value for $L$ into the first condition, we find $\epsilon_{\text{circle}}(r) = 2\epsilon_t(r/\sigma)^2$. This is obviously maximized for the circular orbit at the maximum radius of the trap, $r = \sigma$, so that $\epsilon_{\text{max}} = 2\epsilon_t$.

In summary, we have found that for $\epsilon \in [0, \epsilon_t]$,

$$r_{\text{min}} = 0$$
$$r_{\text{max}}(\epsilon) = \sigma \sqrt{\epsilon/\epsilon_t}$$

(4.4)
and for $\epsilon > \epsilon_t$,

$$
\begin{align*}
\tau_{min}(\epsilon) &= \sigma \sqrt{\frac{\epsilon}{\epsilon_t} - 1} \\
\tau_{max} &= \sigma \\
L^2_{min}(\epsilon) &= 2m\sigma^2(\epsilon - \epsilon_t) \\
L^2_{max}(r, \epsilon) &= 2mr^2(\epsilon - \epsilon_t(r/\sigma)^2),
\end{align*}
$$

(4.5)

where it is understood that the integral over $r$ in Eq. 3.4 simply runs from 0 to $\sigma$ when $\epsilon > \epsilon_t$. Using these limits in equations 3.4 and 3.5, we find

$$
\rho_{\text{trapped}}(\epsilon) = \begin{cases} 
\pi^3 \sqrt{2}(m/\epsilon_t)^{3/2}(\sigma/h)^3 \epsilon^2, & \epsilon \in [0, \epsilon_t] \\
\pi^3 \sqrt{2}(m/\epsilon_t)^{3/2}(\sigma/h)^3(\epsilon - 2\epsilon_t)^2, & \epsilon \in (\epsilon_t, 2\epsilon_t],
\end{cases}
$$

(4.6)

$$
\rho_{\text{all}}(\epsilon) = \begin{cases} 
\pi^3 \sqrt{2}(m/\epsilon_t)^{3/2}(\sigma/h)^3 \epsilon^2, & \epsilon \in [0, \epsilon_t] \\
2\pi^2(2m)^{3/2}(\sigma/h)^3 \sqrt{\epsilon} \\
\times \left(1 - \frac{1}{2}(\epsilon/\epsilon_t)\right) \sqrt{1 - (\epsilon_t/\epsilon)} + \frac{1}{2}(\epsilon/\epsilon_t)^{3/2} \sin^{-1}\left(\sqrt{\epsilon_t/\epsilon}\right), & \epsilon \in (\epsilon_t, 2\epsilon_t].
\end{cases}
$$

(4.7)

Note that $\rho(\epsilon)$ appearing in the kinetic equation is $\rho_{\text{trapped}}(\epsilon)$, since the kinetic equation is concerned with the evolution of atoms in our trapped gas.

Fig. 4.8 and Fig. 4.9 show the density of trapped states and density of all states, respectively.
Figure 4.8: Density of trapped states for harmonic traps. Notice that the density of trapped states is symmetric about $\epsilon = \epsilon_t$, which can be seen from Eq. 4.6 by noting that the function $\rho_{\text{trapped}}(\epsilon)$ in the range $\epsilon > \epsilon_t$ is simply a translation to the right by $2\epsilon_t$ of the parabola defining $\rho_{\text{trapped}}(\epsilon)$ in the range $\epsilon < \epsilon_t$.

Figure 4.9: Density of states for harmonic traps, with untrapped states included

4.2 Limits of integration: the Gaussian trap

Our approach to find the limits of integration in the Gaussian trap will be very similar as for the harmonic trap, with some additional difficulties arising from the more complicated mathematical form of the Gaussian potential function. In the head region $\epsilon \in [0, \epsilon_t]$, a
particle may always exist in the center of the trap so \( r_{\text{min}} = 0 \). The constraint for the upper bound is \( U(r_{\text{max}}) = \epsilon \), which can be solved for \( r_{\text{max}}(\epsilon) = \sigma \sqrt{-\ln(1 - \epsilon/\epsilon_t)} \). For the tail region \( \epsilon > \epsilon_t \), a particle may exist anywhere in the trap, so the integral in Eq. 3.4 runs from 0 to infinity in this region.

As with the harmonic trap, the limits of integration for Eq. 3.5 are more difficult to find in the tail region. The maximum angular momentum passing through radius \( r \) with energy \( \epsilon \) is found in exactly the same way as for the harmonic potential: use the effective potential energy to find the angular momentum corresponding to an orbit with energy \( \epsilon \) and turning point at \( r \). Doing this, we find

\[
L_{\text{max}}^2(r, \epsilon) = 2mr^2(\epsilon - \epsilon_t(1 - e^{-(r/\sigma)^2})).
\]

Plots of \( L_{\text{max}}^2 \) as a function of radius and energy are shown in Fig. 4.10 and Fig. 4.11.

![Figure 4.10: Maximum trappable angular momentum for Gaussian traps as a function of radius. Higher curves correspond to larger energies.](image)

The other three limits of integration are coupled since both \( r_{\text{min}}(\epsilon) \) and \( r_{\text{max}}(\epsilon) \) are extrema of the trajectory with \( L = L_{\text{min}} \). It is not so straightforward to calculate these parameters for the Gaussian trap as it was for the harmonic trap because there is no radius at which we know the effective potential energy function will intersect the line of constant energy. This is because the “trap wall” is at \( r = \infty \), but the effective potential energy function asymptotes to \( \epsilon_t < \epsilon \) as \( r \to \infty \), so any point of intersection between \( U_{\text{eff}}(r) \) and the line of constant energy must occur at some non-obvious point between \( r = 0 \) and infinity.

As before, our strategy is to find the maximum radius of a trapped orbit first and then find the angular momentum that this orbit corresponds to. The maximum radius of a trapped orbit with energy \( \epsilon \) will occur for some turning point so that \( U_{\text{eff}}(r_{\text{max}}, L_{\text{min}}) = \)
Figure 4.11: Maximum trappable angular momentum for Gaussian traps. Steeper slopes correspond to larger radii.

\( \epsilon \). This will serve as our first condition to find \( r_{\text{max}} \).

Now suppose we find some angular momentum \( L_0 \) and position \( r_0 \) such that \( U_{\text{eff}}(r_0, L_0) = \epsilon \). The slope of \( U_{\text{eff}}(r, L_0) \) must be positive, zero, or negative at \( r = r_0 \). If the slope is negative, then the particle will accelerate along \( U_{\text{eff}} \) toward larger radii, and we obviously have \( r_0 \neq r_{\text{max}} \). On the other hand, suppose the slope is positive. Since \( \epsilon > \epsilon_t \) lies above the trap depth on an effective potential energy plot and we know the effective potential energy falls to \( \epsilon_t \) as \( r \to \infty \), there must be a peak in \( U(r, L_0) \) with value greater than \( \epsilon \) at some radius larger than \( r_0 \). Thus if we reduce the angular momentum slightly, this peak will not fall below the line of constant energy \( \epsilon \) and the particle remains trapped. However, the effective potential curve will shift down, allowing the particle to access larger radii, and again we find \( r_0 \neq r_{\text{max}} \). All that remains is the possibility that the slope is zero: \( dU_{\text{eff}}(r)/dr = 0 \). This will serve as another condition to find \( r_{\text{max}} \).

The first condition for a maximum radius, \( \epsilon = U_{\text{eff}}(r) = \epsilon_t(1 - e^{-(r/\sigma)^2}) + L^2/(2mr^2) \), can be solved for \( L^2 = 2mr^2(\epsilon - \epsilon_t(1 - e^{-(r/\sigma)^2})) \). The second condition, \( dU_{\text{eff}}(r)/dr = 2\epsilon_t(r/\sigma^2)e^{-(r/\sigma)^2} - L^2/(mr^3) = 0 \), can be solved for \( L^2 = 2m\epsilon_t(r^4/\sigma^2)e^{-(r/\sigma)^2} \). Equating these two expressions for \( L^2 \) and simplifying, we obtain

\[
e^{-(r/\sigma)^2}((r/\sigma)^2 - 1) = \frac{\epsilon}{\epsilon_t} - 1. \quad (4.8)
\]

Notice that the left-hand side of this equation depends only on \( r \), while the right-
hand side depends only on $\epsilon$. This can be solved for the radii satisfying the conditions we set above, $U_{\text{eff}}(r) = \epsilon$ and $dU_{\text{eff}}(r)/dr = 0$. The solution to this equation involves the product logarithm, alternatively known as the Lambert W-function or omega function. This function is written $W(x)$ and is defined such that $W(x)e^{W(x)} = x$. This is analogous to the ordinary logarithm, which is defined such that $e^{\ln(x)} = x$, hence the name “product logarithm.” I urge the reader to see the excellent Wikipedia article on the “Lambert W function”; for a more detailed, rigorous, and obtuse treatment, see [5].

I claim that the solution to Eq. 4.8 is

$$r = \sigma \sqrt{1 - W((1 - \epsilon/\epsilon_t)e)}. \quad (4.9)$$

By substituting this expression for $r$ into Eq. 4.8, we can simplify the equation to $W((1 - \epsilon/\epsilon_t)e)e^{W((1 - \epsilon/\epsilon_t)e)} = (1 - \epsilon/\epsilon_t)e$, which is true by the definition of the product logarithm. This confirms that $r = \sigma \sqrt{1 - W((1 - \epsilon/\epsilon_t)e)}$ is indeed a solution to Eq. 4.8.

However, we are not done yet. The product logarithm is a multi-valued function and we still must determine the correct order of its solution to use in defining $r_{\text{max}}$. It will be useful to know the maximum energy $\epsilon_{\text{max}}$ of a trapped state so we can determine the range of the argument in $W((1 - \epsilon/\epsilon_t)e)$. By the same argument used to determine the maximum energy of a trapped state in the harmonic trap, we can restrict ourselves to examining circular orbits at each possible radius $r$ and maximize $\epsilon$ with respect to $r$. Just as in the harmonic case, a circular orbit is defined by $\epsilon = U_{\text{eff}}(r)$ together with $dU_{\text{eff}}(r)/dr = 0$. We just found that this second condition implies $L^2 = \frac{2m\epsilon_t}{\sigma^2}(r^4/\sigma^2)e^{-(r/\sigma)^2}$. If we plug this expression for $L^2$ into the first condition, we find $\epsilon_{\text{circle}} = \epsilon_t(1 - e^{-(r/\sigma)^2}(1 - (r/\sigma)^2))$. To maximize this energy, we set the $r$-derivative equal to 0. Solving for $r$, we then find $r = \sqrt{2}\sigma$ is the radius at which a circular orbit has the highest energy. Plugging this value back into the equation for $\epsilon_{\text{circle}}$, we find that $\epsilon_{\text{max}} = \epsilon_t(1 + e^{-2}) \approx 1.14\epsilon_t$.

Therefore, the argument of the product logarithm is no less than $(1 - (1 + e^{-2}))e = -1/e$ and is no more than 0 since $\epsilon > \epsilon_t$ for the region we are considering. There are two real values of the product log for an argument in the range from $-1/e$ to 0, denoted $W_{-1}(x)$ and $W_0(x)$. These values are called the negative-first-order and zeroth-order (or principle) values of the product log, respectively. It would perhaps be unjustified to simply choose whichever value gives a larger radius in Eq. 4.9. Let us then consider why there should be two real solutions to this equation in the first place.

Recall that we found Eq. 4.8 by considering the radius at which the effective potential energy curve intersects the line of constant energy and has a slope of zero (intuitively,
the zero slope means the potential energy curve just barely meets up with the line of constant energy. Notice that this exactly describes the radius of a circular orbit with energy $\epsilon$ since the condition $dU_{\text{eff}}(r)/dr = 0$ implies there is no radial force on the particle. A circular orbit will either be stable or unstable. If the orbit is unstable, then any perturbation might cause the particle to fall back toward lower radii or to fall away to infinity. This is the case that we are looking for. However, we might instead have a stable circular orbit. In this case, we could lower the potential energy curve and extend the accessible range of radii to the particle, so stable circular orbits with energy $\epsilon$ do not occur at radius $r_{\text{max}}(\epsilon)$. Each of these situations is represented in Fig. 4.12.

Notice that for any $\epsilon$ in the range of trappable energies, there is a stable circular orbit with energy $\epsilon$ and an orbit with energy $\epsilon$ that has maximal radius. Since there are two physical situations described by Eq. 4.8 and two solutions in Eq. 4.9 corresponding to different values of the product log, we can be assured that both solutions must represent some physical situation for a trapped state. Thus $r_{\text{max}}(\epsilon)$ is simply the larger solution to Eq. 4.9.

Let us now determine which value of the product log corresponds to larger radii. Each value of the product log is shown in Fig. 4.13. $W_{-1}(-1/e) = -1$ and monotonically decreases to $-\infty$ as the argument approaches zero from below. Thus using $W_{-1}(x)$ for the product log in Eq. 4.9, we find that $r$ ranges from $\sqrt{2}\sigma$ to infinity. On the other hand, $W_{0}(-1/e) = -1$ and monotonically increases to 0 as the argument goes
Figure 4.13: $W_0(x)$ (solid) is greater than $W_{-1}(x)$ (dotted) for all values of $x$ except $x = -1/e$, where they are equal to 0. Therefore, using $W_0(x)$ for the product log in Eq. 4.9, we find that $r$ ranges from $\sigma$ to $\sqrt{2}\sigma$. In conclusion, for all values of the argument, $W_{-1}(x) > W_0(x)$ except for $W_{-1}(-1/e) = W_0(-1/e)$. Therefore, using $W_0(x)$ in Eq. 4.9 corresponds to stable circular orbits with energy $\epsilon$ while using $W_{-1}(x)$ corresponds to the orbits with energy $\epsilon$ that have the largest outer turning points.

Figure 4.14: Maximum trappable radius for Gaussian traps, found at a point of intersection between $U_{eff}(r,L_{min})$ and the line of constant energy $\epsilon$ where the slope of $U_{eff}(r)$ is zero.
Figure 4.15: Minimum trappable angular momentum for Gaussian traps. Notice that unlike for harmonic traps, this function is not linear because \( r_{\text{max}}(\epsilon) \) varies with energy.

At last, we can write the closed-form solution to Eq. 4.8 for the maximal radius of any orbit with energy \( \epsilon \): 

\[
    r_{\text{max}} = \sigma \sqrt{1 - W - 1 \left( 1 - \frac{\epsilon}{\epsilon_t} \right)e}.
\]

This is shown in Fig. 4.14. It is now easy to solve \( \epsilon = U(r_{\text{max}}) \) to find 

\[
    L_{\text{min}}^2(\epsilon) = 2mr_{\text{max}}^2(\epsilon - \epsilon_t(1 - e^{-(r_{\text{max}}/\sigma)^2})).
\]

This is shown in Fig. 4.15.

Figure 4.16: Minimum trappable radius for Gaussian traps, found numerically from the lower point of intersection between \( U_{\text{eff}}(r, L_{\text{min}}) \) and the line of constant energy epsilon.

Unfortunately, I believe it is impossible to find \( r_{\text{min}}(\epsilon) \) analytically. However, it is straightforward to find it numerically, since it is the lower point of intersection between
the effective potential energy curve $U_{\text{eff}}(r, L_{\text{min}})$ and the line of constant energy $\epsilon$. A plot of $r_{\text{min}}(\epsilon)$ found in this way is shown in Fig. 4.16.

In summary, we have found that for $\epsilon \in [0, \epsilon_t]$,\[
\begin{align*}
r_{\text{min}} &= 0 \\
r_{\text{max}}(\epsilon) &= \sigma \sqrt{-\ln(1 - \epsilon/\epsilon_t)}
\end{align*}
\]
and for $\epsilon > \epsilon_t$,
\[
\begin{align*}
r_{\text{max}}(\epsilon) &= \sigma \sqrt{1 - W_{-1}((1 - \epsilon/\epsilon_t)e)} \\
L_{\text{min}}^2(\epsilon) &= 2mr_{\text{max}}^2(\epsilon - \epsilon_t(1 - e^{-(r_{\text{max}}/\sigma)^2})) \\
L_{\text{max}}^2(r, \epsilon) &= 2mr^2(\epsilon - \epsilon_t(1 - e^{-(r/\sigma)^2})),
\end{align*}
\]
where $r_{\text{min}}(\epsilon)$ is found numerically and it is understood that the limits of integration for Eq. 3.4 run from 0 to infinity when $\epsilon > \epsilon_t$.

Obviously, we cannot find a closed form for $\rho_{\text{trapped}}$ in the tail region $\epsilon > \epsilon_t$ since we cannot even analytically find the lower limit of integration on $r$. Further, I cannot find a closed form solution for $\rho_{\text{all}} = \rho_{\text{trapped}}$ in the head region $\epsilon \in [0, \epsilon_t]$. We therefore numerically compute the density of states $\rho_{\text{trapped}}(\epsilon)$ for the Gaussian trap. The density of trapped states is shown in Fig. 4.17. However, we can see that $\rho_{\text{all}}(\epsilon)$ diverges for $\epsilon > \epsilon_t$ since the integral in Eq. 3.4 approaches $\int r^2 dr$ in the limit as $r \to \infty$. Thus $\rho_{\text{trapped}}(\epsilon)/\rho_{\text{all}}(\epsilon) = 0$ for $\epsilon > \epsilon_t$. Intuitively, the atoms are essentially free particles upon escaping to the region arbitrarily far from the center of the trap, and therefore have infinitely many available states.
Figure 4.17: Density of trapped states in a Gaussian trap. The density of all states (including untrapped states) is the same up to the trap depth and then diverges.
Chapter 5

Results and discussion

5.1 Results and analysis

We have found the evolution of the distribution function in Gaussian traps and harmonic traps following the procedure outlined in Ch. 3. Let \( \eta \equiv \epsilon_t / kT \) be the ratio of trap depth to characteristic thermal energy in the gas, and \( \eta_0 \equiv \epsilon_t / kT_0 \) be the value of \( \eta \) when the gas is loaded into the trap. For Gaussian traps, we have considered atoms loaded with integer values of \( \eta_0 \) between 1 and 10, while for harmonic traps we have considered atoms loaded with integer values of \( \eta_0 \) between 1 and 7. For harmonic traps, at \( \eta_0 = 8 \) the evaporation is too slow to balance heating effects and no cooling occurs. The evolution of the distribution function for shallow (small \( \eta_0 \)) and deep (large \( \eta_0 \)) Gaussian and harmonic traps is shown in Fig. 5.2.

The distribution functions evolve for between about 5 and 80 time steps, with short evolutions corresponding to deep traps where heating effects quickly balance cooling due to evaporation. At each of these steps, we record the temperature and number of atoms in the trap. These two parameters fully characterize the gas. See Fig. 5.1 for a characteristic trajectory of temperature as a function of time and number as a function of temperature.

It is a surprisingly subtle and difficult problem to decide on any parameter to characterize how “good” the evaporation is. I begin by discussing an obvious candidate and then expose some of its limitations.

Consider the “efficiency” of evaporation \( \Delta T / \Delta N \), which gives the change in temperature per atom lost. We would ideally like for this parameter to be maximized so that we can obtain a large drop in temperature with a small drop in the number of atoms. The peak values of the efficiency are shown in Fig. 5.3 for every cooling trajectory we
computed. The peak efficiencies usually occur very early in the evaporation process while there are still many high-energy atoms at the upper edge of the distribution function that can easily leave the trap. This observation justifies the experimental practice, known as forced evaporation, of continuously lowering the trap depth.

For both the harmonic and Gaussian potentials, lower trap depths produce more efficient cooling because the exponential decay in $f(\epsilon)$ does not greatly suppress the distribution function for the highest-energy atoms, which are at a lower energy than the corresponding highest-energy atoms in deep traps. Thus most of the atoms lost have energies near $\epsilon_t$ and the average energy of the remaining atoms decreases rapidly. This effect is more pronounced for the harmonic trap with small $\eta_0$ because the tail of the density of states extends to $2\epsilon_t$, whereas the Gaussian potential can only hold atoms with energies about 15% greater than the trap depth.

Conversely, traps with large $\eta$ have few atoms occupying the highest-energy states, so atoms with energies much less than $\epsilon_t$ must be removed to significantly reduce the average energy. Since these atoms are more likely to have energies near the average energy of the gas than those removed in shallow traps, more of them must be removed in order to lower the temperature by a given amount.

For similar reasons, Gaussian traps cool more efficiently at medium and large $\eta$ than harmonic traps. The head of the Gaussian density of states nearly diverges as $\epsilon$ approaches $\epsilon_t$. Thus even for large $\eta$, there is a sharp increase in $\rho(\epsilon)f(\epsilon)$ at the trap depth, which serves a similar function as the large tail of the harmonic trap at small $\eta$. The harmonic trap, on the other hand, has very little cooling power at high trap depths.
Based solely on these considerations, one might conclude that it is best to load into an arbitrarily shallow trap. The obvious problem with this approach is that the number of atoms that get loaded into a trap depends on the trap depth and is generally larger for deeper traps. Thus very shallow traps do not produce large enough samples of atoms to be experimentally viable.

Intuitively, we might expect that for very deep traps $\eta \gg 1$, almost all of the atoms will be in states confined to the bottom of the potential energy curve very near the center of the trap. In this case, we may expand the trapping potential in a Taylor series and neglect high-order terms. Since we chose the coefficient of the harmonic trapping potential such that each trap has the same curvature at $r = 0$, we know that the Taylor expansion of each trapping potential about $r = 0$ is equal to $U(r) = \epsilon_t (r/\sigma)^2 + O(r^3)$. Of course, the harmonic trap does not have higher than second-order terms in its Taylor expansion, so this simply states that a sufficiently deep Gaussian trap behaves like a harmonic trap with the same depth. Thus we would expect the Gaussian and harmonic traps to behave similarly for sufficiently large trap depths. The data in Fig. 5.3 shows that this is not the case up to $\eta = 10$, raising the question of how large a trap depth must be before the approximation of a Gaussian potential as harmonic is justified. Notice that to justify the approximation of a Gaussian trap as harmonic, the number of atoms in spatial regions where the trapping potential is significantly anharmonic must be negligible. However, since any evaporating atom travels out to infinity where the Gaussian trapping potential is completely anharmonic, the approximation of a harmonic trap does not apply well for considering those atoms that will evaporate. Thus the approximation of a Gaussian trap as harmonic is seen to be rather poor when evaporative processes are significant, even for relatively large trap depths.

The last comment I will make about our results is that for very deep traps $\eta \gg 1$, the tails in both traps are damped out so as to appear almost inconsequential. As mentioned before, many groups truncate the distribution function at the trap depth on the assumption that ergodic mixing will result in all high-energy atoms leaving the trap automatically. Whether or not this assumption is valid, the practice of truncating the distribution function at the trap depth more closely approximates the dynamics of non-ergodic deep traps than non-ergodic shallow traps since the occupation of atoms with $\epsilon > \epsilon_t$ is smaller for deep traps. However, we see in Fig. 5.4 that even in traps as deep as $\eta_0 = 9$, the cooling trajectories for an ergodic and non-ergodic trap diverge significantly.
Figure 5.2: The evolution of both low- and medium-depth traps. The distribution functions $\rho(\epsilon)f(\epsilon)$ move down and to the left with time, as atoms are lost and the average energy decreases.

(a) Gaussian trap with $\eta_0 = 1$  (b) Gaussian trap with $\eta_0 = 6$

(c) Harmonic trap with $\eta_0 = 1$  (d) Harmonic trap with $\eta_0 = 6$
Figure 5.3: Peak cooling efficiency for Gaussian and harmonic traps (circles and squares, respectively).

Figure 5.4: Cooling trajectories in ergodic (dashed) and non-ergodic (solid) shallow (left) and deep (right) Gaussian traps. Notice that the horizontal axes have different ranges. For shallow traps, ergodic cooling produces a smaller temperature drop for a given loss in atom number, while for deep traps, ergodic cooling produces a larger drop in temperature for a given loss in atom number. Although the divergence in cooling trajectories is less rapid for the deep trap, the difference between ergodic and non-ergodic cooling is still extremely non-negligible.
5.2 Future work

We now have a functional model of evaporative cooling and a large set of data on cooling trajectories in different isotropic traps, but it is important to develop a more useful metric for evaluating and comparing traps. With this in mind, recall that the important parameter in Bose-Einstein condensate experiments is the phase-space density $\rho_{PS}$ rather than the temperature $T$ (see the Introduction). To evaluate the phase-space density of a gas upon loading into an evaporative cooling trap, one must first estimate the temperature and number density of the loaded gas. Estimating the temperature is typically straightforward: in our lab, the temperature of the compressed magneto-optical trap (CMOT) is currently on the order of 270 $\mu$K. Since we load atoms into the dipole trap directly from the CMOT, this will also be the initial temperature of atoms in the dipole trap. However, the determination in advance of the number of atoms loaded into the dipole trap is far from obvious, even when given the trap depth and the properties of the CMOT. Further, since the trapping potential is not uniform, the number density in the evaporative cooling trap is a complicated function of space and the temperature of the gas. Similar difficulties apply to the harmonic trap.

If we could find the number density and temperature of a gas upon loading into a trap and then find how the number density changes with the temperature and number of remaining atoms at later times, then we could compute the evolution of the phase-space density $\rho_{PS}$ from the data we have already obtained. Once this work is done, I propose that the quality of a trap be evaluated as follows: first, consider only those traps that produce a phase-space density on the order of unity. Among these traps, the one with the largest number of atoms left upon reaching unity phase-space should be considered the best trap. This metric will serve especially well for experiments that seek to examine behavior near the transition point; for other experiments where it is more important to obtain the maximum possible occupation of the ground state, more emphasis might be placed on larger phase-space densities at the expense of emphasis on having a large number of atoms.

In addition to finding a better metric with which to evaluate our data, the model presented here could be extended to trapping potentials with cylindrical symmetry. In principle, the only significant adaptation required to model a cylindrically symmetric trap is that we must find the limits of integration for trapped states in a trap exhibiting cylindrical symmetry; however, as we saw in Ch. 4, determining these limits of integration is non-trivial even in the case of spherical symmetry. Further, there exist dipole traps that nearly achieve spherical symmetry by crossing multiple laser beams, and as
mentioned previously, the TOP trap can be modified to produce a spherically symmetric trapping potential, so the generalization to cylindrical symmetry is not crucial.

We have only considered free evaporation, in which the trapping potential is fixed throughout the evaporation process. One might consider applying this model to forced evaporation, in which the trap depth is continuously lowered.

Perhaps the most important future work on this project will be a comparison of our computational results with experimental data. This should be more or less straightforward once we are consistently dipole trapping atoms.

Anyone interested in building on the results presented here is encouraged to peruse Arolyn Conwill’s undergraduate thesis, as this work is an extension of the research described there [4].

5.3 Conclusions

We develop a model of evaporative cooling to describe the evolution of the distribution function of trapped atoms. One important contribution of this work is that we consider atoms with energies large enough to escape the trapping potential well, but which are in a state where they do not leave the boundary of the trap. We compare evaporation in both isotropic Gaussian and harmonic potentials at a variety of depths compared to the characteristic thermal energy of the loaded gas. Although we have a large amount of data with which to draw a comparison, we still need to develop a more useful metric for evaluating the quality of a trap.
Bibliography


Appendix A

Sample Mathematica Code

Below is the Mathematica code used to compute the evolution of the distribution function for an isotropic Gaussian trapping potential. The code to evolve the distribution function for a harmonic trapping potential is similar.

(*This file is making all of the units consistent with reasonable \ starting values for parameters.*)

\[Epsilon\]t0 = 1(*First trap depth*)
\[Delta\]E = .01
\[Sigma\] = 1(* Trap Width. Defines length unit. About 5E-5 m.*)
m = 1 (*Defines mass unit. About 1.45E-25 kg for Rb-87.*)
h = 1 (*Defines "Energy*Time" or "mass*length^2/time" unit. About \ 6.26E-34 Js.*)
Cst = (2*Pi*(2*m)^(3/2))/(h)^3
\[Theta\]min = 0
\[Theta\]max = 2*Pi
\[Phi\]min = 0
\[Phi\]max = Pi
n0 = .01
k = 1 (*Defines Temperature unit to be the same as Energy unit, AKA \n(units of h)\(^2/(\text{mass} \times \text{length})^2\). About 1.38E-23 J/K.*)
T0 = 5*10^-6 (*Initial temperature corresponding to about 439 \microKelvin.*)

a = .0001 (*Must be related in a reasonable way to \(\sigma\). 1/10,000 \seems about right.*)

\[ f_0[\epsilon, n, T] = n \times e^{-\frac{\epsilon}{kT}} \text{ (* Boltzmann distribution function *)} \]

\[ \epsilon_{3\text{step}}[y] = \text{Piecewise}[\{1, y \geq 0\}, \{0, y < 0\}] \text{ (* Heavyside step function *)} \]

ndata = 100

(*Needs to be given a value*)
Heating = 1.6*10^-7

(*Arbitrary, but should be relatively small.*)
LossRate = 10^5

(*Go through all commands multiple times, changing the trap depth \each run.*)
For[runthrough = 0, runthrough <= 1, runthrough++,

\[ \epsilon_t = (\epsilon_t + \text{runthrough}) \times kT_0, \]

timesteps = 500,

U0 = \(\epsilon_t\), (* Trap Depth *)

\[ \Theta[x] = \text{Piecewise}[\{1, 0 < x < \epsilon_t (1 + \exp[-2])\}, \{0, \epsilon_t (1 + \exp[-2]) \geq 0\}] \text{ (*}
Heavyside step function *),

\[U_3[r, \[Theta], \[Phi]] = U_0(1 - \text{Exp}[-r^2]),\]

(*Potential of the trap*)

Print[Plot[U3[r, 0, 0], {r, -3, 3}]],

(*Limits of integration for both the head and the tail of the density of states*)

\[r_{\text{max head}}[\[Epsilon]] = \[Sigma] \sqrt{-\text{Log}[(1 - \[Epsilon]/U_0)]};\]
\[r_{\text{min head}}[\[Epsilon]] = 0;\]

\[r_{\text{max tail}}[\[Epsilon]] = \sqrt{1 - \text{ProductLog}[-1, E(\[Epsilon]_t - \[Epsilon])/(\[Epsilon]_t)]];\]

\[L_{\text{Squared min}}[\[Epsilon]] = 2m (r_{\text{max tail}}[\[Epsilon]]^2) (\[Epsilon] - \[Epsilon]_t (1 - \text{Exp}[-r_{\text{max tail}}[\[Epsilon]]^2]));\]

\[L_{\text{Squared max}}[r, \[Epsilon]] = 2m*r^2 (\[Epsilon] - \[Epsilon]_t (1 - \text{Exp}[-r^2]));\]

\[r_{\text{min tail}}[\[Epsilon]] := r /. \text{FindRoot}[\[Epsilon] == L_{\text{Squared min}}[\[Epsilon]]/(2m*r^2) + \[Epsilon]_t (1 - \text{Exp}[-r^2]), \{r, .2\}];\]

(*The density of states calculated exactly for the head and tail*)

\[\[\text{Rho}]\text{exact}[\[Epsilon]] := \text{Piecewise}[\{\]
\{Cst* N\text{Integrate}[
\text{Sqrt}[\[Epsilon] - U_3[r, 0, 0]]*r^2*\text{Sin}[\[Phi]], \{r, r_{\text{min head}}[\[Epsilon]], r_{\text{max head}}[\[Epsilon]]\}, \{\[Theta], \[Theta]_{\text{min}}, \[Theta]_{\text{max}}\}, \{\[Phi], \[Phi]_{\text{min}}, \[Phi]_{\text{max}}\}, \[Epsilon] <= \[Epsilon]_t\},\]
\{((\[Pi]/h)^3)*4/\[Pi]*\text{Sqrt}[2m]*
N\text{Integrate}[\]

III
\(1/\sqrt{\varepsilon - L^2/(2 m r^2) - U_3[r, 0, 0]}\), \(\{r, \\\text{rmintail}[\varepsilon], \text{rmaxtail}[\varepsilon]\}\), \(\{L^2, L^2_{\text{min}}[\varepsilon], L^2_{\text{max}}[\varepsilon]\}\), \(\varepsilon_t < \varepsilon < \varepsilon_t (1 + \exp[-2])\)

(*Print[Plot[\[Rho]exact[\[Epsilon]],{\[Epsilon],0,\[Epsilon]t(1+\exp[-2])}],]*)

(* FORCED REAL *)

(*Take sample points for an interpolating function of the density of states*)
segments = 100,
segmentInterval = (1 + Exp[-2])\[Epsilon]t/segments,
densitydata = Table[\{\[Epsilon], Re[\[Rho]exact[\[Epsilon]] // N]\}, \{\[Epsilon], 0, \[Epsilon]t (1 + \exp[-2]) - \text{segmentInterval}, \text{segmentInterval}\}],
Print[densitydata],
\(f[\[Epsilon]_, n_, T_] = f0[\[Epsilon], n, T]\text{\[CapitalTheta][\[Epsilon]] (* Boltzmann distribution multiplied by step function *)},\)

(*Plot[GravityTruncation[\[Epsilon]],{\[Epsilon],0,2*U0}]*)
\(\[Rho] := \text{Interpolation}[\text{densitydata}],\)
Print["\[Rho]"],
Print[Plot[\[Rho][\[Epsilon]], {\[Epsilon], 0, \[Epsilon]t (1 + \exp[-2])}, \text{Filling} -> \text{Axis}, \text{PlotRange} -> \text{All}]],
\(\[Rho]f[\[Epsilon]_, n_, T_] = \[Rho][\[Epsilon]]f[\[Epsilon], n, T], (* Truncated Boltzmann distribution times density of states *)\)

IV
(*The kinetic equation*)

\[
\text{integrand}[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, n, T] = \\
\text{With}[\{\varepsilon_3 = \varepsilon_2 + \varepsilon_1 - \varepsilon_4\}, (\\\rho_{\text{Min}}[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]) \times\\n\text{Piecewise}[\{(1, \varepsilon_4 \leq \varepsilon_t), (0, \varepsilon_4 > \varepsilon_t)\} * f[\varepsilon_1, n, T] f[\varepsilon_2, n, T] - \\
f[\varepsilon_3, n, T] f[\varepsilon_4, n, T] ) * \varepsilon_3^{\text{step}[\varepsilon_3]},
\]

(* Change in $\rho f$ with respect to time *)

\[
\rho f_{\text{dot}}[\varepsilon_4, n, T] := \frac{m^8 \pi a^2}{(\pi^2 (h/(2 \pi)^3))} N\text{Integrate}[\\n\text{integrand}[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, n, T], \{(\varepsilon_1, 0, 2 \varepsilon_t (1 + \text{Exp}[-2])), \{(\varepsilon_2, 0, \varepsilon_1 - \varepsilon_4)\}],
\]

\[
\rho f_{\text{new}}[\varepsilon, n, T] := \rho[\varepsilon] f[\varepsilon, n, T] + \rho f_{\text{dot}}[\varepsilon, n, T] * \delta t \rho \text{CapitalTheta}[\varepsilon],
\]

(*Initialize values*)

ncurrent = n0, 
Tcurrent = T0, 
nevolution = \{n0\}, 
Tevolution = \{1\}, 
Num = \{1\}, 
NumCurrent = 1, 
unscaledN0 = N\text{Integrate}[\\n\text{ncurrent} \rho[\varepsilon] \text{Exp}[-\varepsilon/(k \times T0)], \{(\varepsilon, 0, \varepsilon_t (1 + \text{Exp}[-2]))],
\]

Print["Number: ", unscaledN0],

(* Calculate the initial average energy*)

V
Eavg0 = NIntegrate[\[Epsilon]*\[Rho]f[\[Epsilon], ncurrent, Tcurrent], {\[Epsilon], 0, \[Epsilon]t (1 + Exp[-2])}], Print[Eavg0], Eavg = {1},

ETotal0 = NIntegrate[\[Epsilon]*\[Rho]f[\[Epsilon], ncurrent, Tcurrent], {\[Epsilon], 0, \[Epsilon]t (1 + Exp[-2])}], Print[ETotal0], ETotalCurrent = 1, ETotal = {ETotalCurrent},

DeltaE = {}, DeltaN = {}, DeltaEavg = {}, DeltaT = {},

(*Create an interpolating function to find the temperature from average energy*)
AvgAtOne = NIntegrate[\[Epsilon]*\[Rho][Epsilon]*Exp[-\[Epsilon]/(k*T0)], {\[Epsilon], 0, \[Epsilon]t (1 + Exp[-2])}]/NIntegrate[\[Rho][Epsilon]*Exp[-\[Epsilon]/(k*T0)], {\[Epsilon], 0, \[Epsilon]t (1 + Exp[-2])}],
AverageEnergyExact[T_] := NIntegrate[\[Epsilon]*\[Rho][Epsilon]*Exp[-\[Epsilon]/(k*T)], {\[Epsilon], 0, \[Epsilon]t (1 + Exp[-2])}]/NIntegrate[\[Rho][Epsilon]*Exp[-\[Epsilon]/(k*T)], {\[Epsilon], 0, \[Epsilon]t (1 + Exp[-2])}]/AvgAtOne,
AvgEnergydata = Table[{T, AverageEnergyExact[T]}, {T, 0.01*T0, 1*T0, .01*T0}],

VI
Print[AvgEnergydata],
AverageEnergy := Interpolation[AvgEnergydata],
Print[Plot[AverageEnergy[T], {T, 0, T0}]],

Print["\[Rho]f"],
Print[Plot[\[Rho]f[\[Epsilon], ncurrent, Tcurrent], {\[Epsilon],
0, \[Epsilon]t (1 + Exp[-2])}],
i = 1,
continue = True,
(*fcurrent and \[Rho]fdotnext are unnecessary steps,
and appear to be there only to generate graphs.*)
While[continue,
{i++,
Print[\[Epsilon]t, " - ", i],
(*Print["fcurrent",i],*)
(*fcurrent[\[Epsilon]_] = f[\[Epsilon], ncurrent, Tcurrent],*)
(*Print[Plot[fcurrent[\[Epsilon]], \[Epsilon]t]],*)
(*Print["\[Rho]fcurrent",i],*)
(*\[Rho]fcurrent[\[Epsilon]_] = \[Rho]f[\[Epsilon], ncurrent, Tcurrent],*)
(*Print[Plot[\[Rho]fcurrent[\[Epsilon]], \[Epsilon]t]],*)
Print["\[Rho]fdotnext", i],
integrand[\[Epsilon]1, \[Epsilon]2, \[Epsilon]3, \[Epsilon]4, ncurrent, Tcurrent],
\[Rho]fdotnext[\[Epsilon]4_] :=
NIntegrate[
VII
\[\text{integrandnext}[\[Epsilon\]1, \[Epsilon\]2, \[Epsilon\]3, \[Epsilon\]4], \{\[Epsilon\]1, 0, 2*\[Epsilon\]t (1 + \text{Exp}[-2])}, \{\[Epsilon\]2, 0, \[Epsilon\]1 - \[Epsilon\]4}\] - \text{LossRate} \times \[\text{Rho}\]f[\[Epsilon\]4, ncurrent, Tcurrent],

\text{(*Print}\\[\text{Plot}[\[\text{Rho}\]fdotnext[\[Epsilon\]],\{\[Epsilon\], 0, \[Epsilon\]t(1+\text{Exp}[-2])},\text{PlotRange}\rightarrow\text{All}],*)

\text{(*Calculate the rate of cooling*)}
\text{Edot} = \text{NIntegrate}[\[\text{Rho}\]fdotnext[\[Epsilon\]]\times\[Epsilon\], \{\[Epsilon\], 0, \[Epsilon\]t (1 + \text{Exp}[-2])}] / \text{ETotal0},

\text{Ndot} = \text{NIntegrate}[\[\text{Rho}\]fdotnext[\[Epsilon\]], \{\[Epsilon\], 0, \[Epsilon\]t (1 + \text{Exp}[-2])}] / \text{unscaledN0},

\text{(*Make sure the system is still losing temperature*)}
\text{If} [\text{Heating} \times \text{NumCurrent} + \text{Edot} \geq \text{Ndot} \times \text{ETotalCurrent} / \text{NumCurrent}, \text{continue} = \text{False}],

\text{(*If the system is in equilibrium, stop.*)}
\text{If} [k \times \text{Tcurrent} > 0.2 \times k \times \text{T0} \quad \& \quad \text{continue},
\{\text{deltat} = -[\text{Delta}E] / (\text{Edot} + \text{Heating} \times \text{NumCurrent}),
\text{DeltaNcurrent} = \text{deltat} \times \text{Ndot},
\text{AppendTo} [\text{DeltaT}, \text{deltat}],
\text{AppendTo} [\text{DeltaN}, \text{DeltaNcurrent}],
\text{NumNew} = \text{NumCurrent} + \text{DeltaNcurrent},
\text{Enew} = \text{ETotalCurrent} - [\text{Delta}E],
\text{(*Print}\\[\text{NumNew}: "],\text{NumNew}],
\text{Print}\\[\text{ENew}: "],\text{Enew}],*)
\text{AppendTo} [\text{Num}, \text{NumNew}],
\text{AppendTo} [\text{ETotal}, \text{Enew}],

\text{(*DeltaEavgcurrent}=(\text{NumCurrent}(\text{DeltaEcurrent})-
\text{ETotalCurrent}(\text{DeltaNcurrent})) / (\text{NumCurrent} \times \text{NumNew}),

VIII


```math

AppendTo[DeltaEavg, DeltaEavgcurrent, *)

EavgNew = (Enew)/(NumNew),
AppendTo[Eavg, EavgNew],

Tnew = T /. FindRoot[AverageEnergy[T] - EavgNew, {T, Tcurrent}],
AppendTo[Tevolution, Tnew/T0],
(*Print["Tnew: ", Tnew, *]

nNew =
NumNew*unscaledN0/
NIntegrate[Rho[Epsilon]*
   Exp[-Epsilon/k*Tnew], {Epsilon, 0, Epsilon t (1 + Exp[-2])}],
AppendTo[nevolution, nNew],
(*Print[Plot[Rho[Epsilon],nNew,Tnew],{Epsilon, 0,Epsilon t(1+Exp[-2])}],*)

ncurrent = nNew,
Tcurrent = Tnew,
NumCurrent = NumNew,
ETotalCurrent = Enew,
Print[ncurrent, ", ", NumCurrent, ", ", Tcurrent/T0],

{continue = False,
i = i - 1}
],

Print[nevolution],
Print[Num],
Print[Tevolution],
Print[Eavg],
Print[DeltaT],

TimeTable = {},
```

IX
For\[j = 1, j < i, j++,
    AppendTo[TimeTable, Sum[DeltaT[[n]], {n, j}]]
]

EvolutionTable = {{0, 1, T0, 1}},
For\[j = 1, j < i, j++,
    AppendTo[
        EvolutionTable, {TimeTable[[j]], Num[[j]], Tevolution[[j]]}]
    Print[EvolutionTable],
]

NumTime = {},
For\[j = 1, j < i, j++,
    AppendTo[NumTime, {TimeTable[[j]], Num[[j]]}]
    Print[Show[
        ListPlot[NumTime, Frame -> True,
            FrameLabel -> {"Time (Arb. units)", Subscript["N/N", 0]},
            LabelStyle -> Directive[Bold, FontFamily -> "Helvetica", Large],
            PlotRange -> Full]]],
]

TempTime = {},
For\[j = 1, j < i, j++,
    AppendTo[TempTime, {TimeTable[[j]], Tevolution[[j]]}]
    Print[Show[
        ListPlot[TempTime, Frame -> True,
            FrameLabel -> {"Time (Arb. units)", Subscript["T/T", 0]},
            LabelStyle -> Directive[Bold, FontFamily -> "Helvetica", Large],
            PlotRange -> All]]],

NumTemp = {},
For\[j = 1, j < i, j++ ,
    AppendTo[NumTemp, {Tevolution[[j]], Num[[j]]}]
    Print[Show[
        ListPlot[NumTemp, PlotRange -> Full, Frame -> True,
            FrameLabel -> {Subscript["T/T", 0], Subscript["N/N", 0]},
            LabelStyle -> Directive[Bold, FontFamily -> "Helvetica", Large],
            PlotRange -> All]]],

X
Snorkelfinch = {},
For[j = 1, j <= i, j++,
  AppendTo[Snorkelfinch,
    Plot[\[Rho]\[Epsilon]*nevolution[[j]]*
      Exp[-\[Epsilon]/(Tevolution[[j]]*T0)] , {\[Epsilon],
        0, \[Epsilon]t (1 + Exp[-2])},
      PlotStyle -> {Thick,
        RGBColor[
          Cos[Sin[Sin[Sin[Sin[j*\[Pi]/(2*i)]*\[Pi]/2]*\[Pi]/2]*\[Pi]/2]/
            2]*\[Pi]/2, 0,
          Sin[
            Sin[Sin[Sin[j*\[Pi]/(2*i)]*\[Pi]/2]*\[Pi]/2]*\[Pi]/2]*\[Pi]/
            2]*\[Pi]/2, .75]}, Frame -> True,
        FrameLabel -> {Subscript["E/kT", 0], "Occupation (Arb. units)"},
        LabelStyle -> Directive[Bold, FontFamily -> "Helvetica", Large],
        PlotRange -> All},
    ]
  ]
  ];
Print[Show[Snorkelfinch]]
]