Tidal Deformations of Sea Ice in Shallow Arctic Regions

A thesis submitted in partial fulfillment of the requirements of a degree of Bachelor of Arts in Physics and Astronomy at Pomona College

Rylan Grady

with advisors
Alma Zook, Ph.D.
Professor, Pomona College

and
Aleksey Marchenko, Ph.D.
Professor, University Centre in Svalbard (UNIS)

April 23, 2013
Contents

Abstract 4

1 Introduction 5
   1.1 Sea Ice Properties 8
   1.2 The Ice of Sveasundet 10

2 Theory 13
   2.1 Physical Meaning of Stress and Strain (1D) 14
   2.2 Physical Meaning of Stress and Strain (3D) 17
   2.3 General Assumptions 21
   2.4 Thin Plate Small Deflection Theory 23
   2.5 Thin Plate Large Deflection Theory 25
   2.6 The Mathematics of the Stretch Terms 28
   2.7 Stretch Terms in 1 Dimension 29

3 Experimental Methods 33
   3.1 Running the Scanner 34
   3.2 Scanner Limitations 35
3.3 Manipulating the Data ............................................ 37
3.4 Pressure Measurements .......................................... 38

4 Results and Analysis ............................................. 40
4.1 Pressure Sensor Measurements ................................. 40
4.2 Elastic Modulus .................................................. 41
4.3 Small Deformation Theory ........................................ 42
4.4 Large Deformation Theory ........................................ 47
4.5 Discrepancies Between the Two Theories ..................... 51
4.6 The Impact of the Stretch Terms ............................... 52
4.7 Morphological Observations ..................................... 55

5 Conclusion .......................................................... 59
5.1 Further Lines of Inquiry .......................................... 60

6 Appendices .......................................................... 61
6.1 MATLAB Code .................................................. 61
6.2 Additional Figures ............................................... 84

List of Figures .......................................................... 92
Abstract

A terrestrial laser scanner, the RIEGL VZ-1000, was used to record the surface of a rising sheet of sea ice in Svea, Svalbard, Norway from low to high tide on March 23-24 2012. A simple pressure sensor was embedded within the 1.3-meter thick ice sheet on March 24. The resulting position data from the scanner was used to solve for the internal stress state by utilizing two simple-plate bending models from elasticity theory. This paper is intended to evaluate the efficacy of optical measurements for swiftly obtaining order-of-magnitude estimates for coastal sea ice stresses.

Two theories were applied in calculations: the thin plate small deformation and thin plate large deformation theories. The large deformation theory predicted peak normal stress values of 200 kPa occurring at high tide. The small deformation theory predicted peak stresses that were orders of magnitude too large to be considered physically meaningful. Both theories predicted stresses in the tens of megapascals when applied to the crack-laden geometry of the sea ice hinge zone. The pressure sensor recorded stresses of 330 kPa with less than 1% fluctuation over the tidal cycle. The results of both the large deformation theory and the direct pressure measurements are within the range of measured stresses for coastal sea ice in the literature, supporting the use of laser measurements to determine sea ice stresses.
1 Introduction

As the extent of polar sea ice decreases due to global warming, Arctic waterways are presenting new opportunities for science and industry. The prospect of untapped mineral and hydrocarbon reserves has already spurred considerable investment into offshore structures and large icebreaking vessels, and the Norwegian oil and gas company Statoil recently made headlines when it pledged to triple its Arctic research budget to $43 million in 2013.

![Figure 1: Minimum extent of Arctic sea ice in 2012 with 30 year average highlighted. (NASA)](image)

Meanwhile, publicly funded Artic research is also booming. Geologists collect ice cores from Greenland’s coastal shelf to study climatic precedents for our planet’s current warming trajectory. Biologists monitor the new competition between temperate water marine life and native arctic species, and proponents of geo-engineering make optical measurements of ice on
land and sea to determine the potential for artificially increasing albedo and helping to slow the pace of global warming.

All of these ventures require coastal infrastructure, and in polar regions, water-facing structures must withstand the destructive action of sea ice. Swiftly moving ridges can fail in crushing modes (Sodhi [2003]), transmitting tremendous forces, and the hinging of land-fast ice due to tidal pressure results in load cycles that threaten infrastructure with premature failure and fatigue (Yue Qian-jin and Yan [2004]). In support of this new age of Arctic exploration and development, engineers are being asked to design for some of the world’s harshest conditions, and, in order to do so, they need accurate parameters. They need to know the magnitude and variability of coastal ice forces.

The purpose of this thesis is to evaluate the effectiveness and feasibility of a terrestrial laser scanning system for determining the internal stresses of coastal sea ice. In the following chapters, I will be using elasticity theory to calculate stresses from measurements of displacements made by the laser scanner, neglecting the effects of sea ice creep. The resulting data will then be compared with measurements made in a more traditional (and onerous) fashion, by embedding a pressure sensor within the deforming ice sheet. In doing so I will consider what place, if any, a laser scanning system has in the site investigation process for planned coastal infrastructure.

The RIEGL VZ-1000 laser scanner used in this research is part of the latest wave of increasingly sophisticated systems used to quantify the risk of
volatile ice floes quickly and cheaply. From the early laser profiles of arctic ice packs (Hibler [1972]), which sought to characterize general conditions, to the unmanned submersibles and aerial drones now being designed to seek out and identify specific ice threats (Zhou and Bachmayer [2011]), these methods rely on optical measurements of size and shape. Apart from the geometry, the most relevant data points have historically been velocities: the speed of the ice floe, the speed of the underlying ocean currents, and the wind speed. However, optical measurements have seldom been used to record deformations in land-fast ice over time.

More broadly, the mechanics of coastal sea ice in general have not received major scientific attention. Some studies have looked at ice pile-up on rubble (Wuebben [1995]; Kovacs and Sodhi [1981]), but a brief survey of the literature yields no attempts at solving for the internal forces of hinging coastal ice from the cyclical deformations of the tide. In this thesis I have relied upon site-specific data and infrastructure put in place by Fabrice Caline during Ph.D. research from 2007-2010. While Caline’s work is arguably among the most comprehensive of coastal sea ice surveys ever published, it did not employ optical measurements for determining stresses. My own research is therefore concerned with bridging two nascent sub-fields of ice mechanics: the use of automated laser measurement systems for determining ice properties and the study of coastal sea ice interactions.
1.1 Sea Ice Properties

One of the greatest challenges facing those who work with sea ice is the fact that nearly every relevant mechanical property of the material is observed to vary over the natural range of temperatures and porosities with which it is encountered in the field (Weeks [2010]). Because sea ice is always found in thermal contact with a reservoir of ocean water beneath, its temperature is held near the freezing point, and in this regime the effects of natural material variations on key properties like elastic modulus and tensile strength are amplified.

![Diagram showing temperature of maximum density, $T_{\text{max}}$, and freezing temperature, $T_f$, vs salinity (in ppt) for sea ice. The change in slope of the $T_f$ line at -8.2°C occurs due to precipitation of Na$_2$SO$_4$10H$_2$O.](image)

The crystal lattice formed by freezing water is highly exclusive. On solidification, impurities are forced out of the tetrahedral $H_2O$ structure...
and into the seawater below. The combination of lower temperature and higher brine density in the seawater nearer the surface then spurs convective mixing, and the downward growth of sea ice begins to follow the familiar time $\frac{1}{2}$ dependence of the Stefan equation. However, due to freezing point depression (see Figure 2 above), macroscopic tubes of brine penetrate into the sea ice surface and can persist even in multi-year ice floes (Schwerdtfeger [1963]).

![Brine tubes at the interface between water and sea ice.](image)

When modeling any large-scale sea ice interaction, it is standard practice to absorb irregularities like brine channels into more general parameters. Density and salinity tests may be performed on individual sea ice cores which are then assumed to be representative samples. Such tests are common in a range of applications from sea ice bearing capacity to determining the mechanical properties of pressure ridges and the magnitudes of ice forces on offshore or coastal structures. Moving from field measurements to engineering material parameters is no small task, but the effects of variations in many sea ice properties are qualitatively simple and straightforward.
One can readily intuit the result of variations in salinity, temperature, and porosity by imagining sea ice on a continuum that includes brittle solids, plastic solids, and liquids. Warm ice is weaker than cold ice because higher temperatures shift mechanical properties towards those of water. For the same reason, a lower salinity value corresponds to a stronger and more brittle material, and freshwater ice is significantly stiffer than first-year sea ice. Higher volumes of brine and air pockets are associated with more plastic behavior.

For the purpose of following this thesis, a qualitative understanding of these effects is sufficient. Because my calculations of internal forces are based in elasticity theory, the key parameter is the elastic modulus, $E$. This value, unlike the Poisson ratio $\nu$, which relates axial to transverse strain, is highly sensitive to the sea ice properties discussed above and is known to vary between 1.5 and 3.0 GPa (Weeks [2010]). In estimating this value I have appealed both to data collected concurrently with my measurements and to site-specific data from previous winters. For an in-depth discussion on the elastic modulus, see the Analysis chapter.

1.2 The Ice of Sveasundet

On the main island of Svalbard, the coal-mining company Store-Norske is interested in expanding its operations. To do so, a bridge or causeway must be built across a 690m-wide tidal inlet to allow the transportation of heavy machinery to new deposits of coal. In 2007, the company co-funded a Ph.D.
position with the University Centre in Svalbard (UNIS) to investigate the action of coastal sea ice on a potential causeway across the inlet, Sveasundet. A 50x25m breakwater was constructed and a small research cabin installed. For three years, Ph D. student Fabrice Caline meticulously documented the ice surrounding the breakwater and experimented with anti-erosion measures to preserve the coastal soil. In the meantime, UNIS has continued to conduct field research in Sveasundet and the surrounding fjord.

The water surface of Sveasundet is covered by sea ice from December to June of each year. From January onwards, the land-fast ice is quite stable and is subject to tidal displacements of around 1m. Due to these displacements a hinge zone forms. Cracks develop between the landfast ice and the floating sea ice that are generally open at low tide and closed at
high tide. An underwater ridge parallel to the fjord’s direction divides the inlet into two 3-5m deep channels and has been known to cause additional cracking of the sea ice when the thickness exceeds the depth of the ridge (Caline [2010]).

![Figure 5: Aerial view of the breakwater at Sveasundet, June 2007. (UNIS)](image)

During the UNIS expedition that included the laser measurements made for this thesis, temperature-density-salinity profiles were made of the ice of Sveasundet, and compression and flexural tests were conducted a few kilometers down the fjord. In addition, three thickness profiles of the ice directly across from the breakwater were made. This information, combined with maps of the seafloor made by Caline, provides sufficient information to develop a theoretical model for determining stresses from measured displacements in the coastal sea ice.
2 Theory

The field of continuum mechanics is concerned with relating applied loads to the internal forces and deformations in a continuous body. Two tensor quantities, stress and strain, are important for quantifying deformations and determining material failure. In this section, I will introduce the topic in one dimension before generalizing and making connections with the physics of simply supported plates and shells.

In practice, engineers and material scientists often begin with a design load and then use the techniques of continuum mechanics to simulate the response of a structure or mechanical component. By adjusting geometric or material parameters, they refine the design to meet the desired tolerances for yield or maximum displacement (stiffness). Measured displacements can also be used to calculate internal forces by applying the same methods in reverse. Doing so is known as solving an inverse problem. See Figure 6.

Figure 6: Engineers and scientists rely upon constitutive laws (Hooke’s Law and its variants) and geometric compatibility to connect internal forces with displacements.
2.1 Physical Meaning of Stress and Strain (1D)

One of the principal challenges in applying the framework of continuum mechanics to a given situation is maintaining the link between the abstract mathematical tensors of stress and strain and the physics of a continuous body. I will begin with an example from elementary physics.

![Figure 7: A rod of cross-sectional area $A$ is loaded uniaxially in tension.](image)

A rod or bar loaded along its axis behaves very much like a spring. When the load is uniformly distributed and the material is homogeneous, the internal forces are constant throughout the rod. The magnitude of this force is represented by the stress, $\sigma = \frac{F}{A}$, which is simply the distributed load over the cross-sectional area with the sign convention that positive stresses are tensile and negative stresses are compressive (Rossmann and Dym. [2008]).

Due to these internal forces, the length of a uniaxially loaded body will change. For small changes in length, the fractional increase or decrease is represented by the strain $\epsilon$ and is given by:

$$\epsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0}$$  \hspace{1cm} (1)
Notice that tensile strains are positive, maintaining the sign convention established for stress. Assuming the density of the material remains constant, conservation of matter requires that the cross-sectional area change inversely with the length. Again, for small changes in length the effect of this reduced area (known as necking) on stress can be neglected, resulting in an expression for the constitutive equation Hooke’s Law in terms of stress and strain:

\[ \sigma = E \epsilon \]  

(2)

where \( E \) is the Young’s Modulus, a material parameter. Now suppose that we are selecting a rod to transmit loads from one mechanical component to another. Using Hooke’s Law we have already determined the strain as a function of \( E \) and \( A \), but what we now need is a relationship involving material displacements. When the rod changes length during tensile loading, some of its original volume elements must move to fill the new space, while others must move to maintain the homogeneity of the material. Given a strain \( \epsilon(x) \), there must be a corresponding displacement function \( u(x) \) that contains the information of where to send each infinitesimal rod element of length \( dx \) to preserve the continuity of the system. To move from strain to displacement we need to develop a compatibility equation.

In Figure 8 below, the rod is divided into 10 equal segments of length \( l_0 \) and the strain is calculated at each segment as an approximation of \( \epsilon(x) \). Given that the strain is constant in this example, you may recognize that
the discretization is trivial, however the formalism used here is directly applicable to higher dimensions. Note that the new length of each segment is given by $l_i = \frac{\sigma}{E}l_0$.

\[ l_0 \quad l_0 \quad \cdots \quad l_0 \]

\[ l_1, l_2 \]

\[ \varepsilon_i \]

\[ u(x) \]

\[ \chi \]

Figure 8: In the one-dimensional case, we can divide the material into intervals of equal length and calculate the strain in each. Geometric compatibility gives us a requirement for the corresponding displacement function.

Consider now the boundaries between segments. The displacement function must send the material that is between the fifth and sixth original segments to a new x-coordinate that is between the fifth and sixth elongated segments. We might write for the material at the $n$th segment, $u(n \cdot l_0) = \frac{\sigma}{E}(n \cdot l_0)$. The farther down the length of the rod we go, the more the displacement function must ‘push’ material to get it to its proper
position. Here we have chosen implicitly a boundary condition, \( u(0) = 0 \), which is reasonable for a uniaxial stress test. In fact, it can be shown that (see Rossmann and Dym. [2008]) this relation holds for arbitrary \( \epsilon(x) \) in one dimension:

\[
\epsilon(x) = \frac{\partial u(x)}{\partial x}
\] (3)

Given this information, we can now select the correct rod material and cross-section for our needs. Because compatibility equations and constitutive laws go both ways, we could use the same techniques to determine a uniaxial load that is applied to a known rod by measuring the displacements.

To summarize:

Figure 9: One-dimensional compatibility equation and constitutive law.

2.2 Physical Meaning of Stress and Strain (3D)

When we move to higher dimensions, no one single number sufficiently describes the internal forces at a point. In fact, because we can have one
component of force perpendicular to and two components of forces parallel to the planes defined by each cartesian coordinate, there are a total of nine different components that contribute to the internal force or stress tensor at every point in a continuous body (See Figure 10 below). Fortunately, rotational equilibrium requires that the \( i \) force acting on the \( j \) area is equal to the \( j \) force acting on the \( i \) area. The nine possible forces on an infinitesimal volume element can therefore be described by a symmetric second-order tensor.

![Figure 10: Stress components acting on an infinitesimal cube.](image)

As components of a tensor, these internal forces will change under trans-
formation of coordinate system. The relative magnitude of normal (diag-
onal) and shear (off-diagonal components) will also vary with coordinate
system. Extracting the physically relevant information, such as whether a
material will fail given critical values for tension or shear, amounts then to
finding the coordinate system in which diagonal or off-diagonal components
are maximized. This is an eigenvalue problem.

Strain is also represented by a symmetric, rank two tensor. While the di-
agonal elements of this tensor continue to take on the same meaning in three
dimensions as the strain scalar $\epsilon$ did in one dimension, the off-diagonal com-
ponents now relate to changes in angle between two previously orthogonal
lengths, as illustrated in Figure 11.

Figure 11: The strain tensor components are related to changes in length
and angle of a deforming infinitesimal cube. For clarity, in this figure and
in Equation 4 below, only one non-zero off-diagonal component has been
included.
\[ \epsilon_{xx} = \frac{L'_x - L_x}{L_x}, \quad \epsilon_{yy} = \frac{L'_y - L_y}{L_y}, \]
\[ \epsilon_{zz} = \frac{L'_z - L_z}{L_z}, \quad \epsilon_{xz} = \epsilon_{zx} = \frac{\theta_{xz}}{2}, \]
\[ \epsilon_{xy} = \epsilon_{yx} = \epsilon_{yz} = \epsilon_{yz} = 0 \]  \hspace{1cm} (4)

But how are 3-dimensional stresses and strains related to one another? We need a more generalized constituent law. In searching for the 3-D analog of Hooke’s Law, we might begin with the rank of our tensors. When we had two scalars, the simplest relationship was that of a constant. See Equation 2 reprinted below.

\[ \sigma = E\epsilon \] \hspace{1cm} (2 Reprinted)

Now, if we were to maintain the sigma term on the left side and the epsilon term on the right, we would need to add something that allowed for the possibility of any component of \( \sigma_{ij} \) influencing any component of \( \epsilon_{ij} \). For this, we need a 4th rank tensor.

\[ \sigma_{ij} = C_{ijkl}\epsilon_{kl} \] \hspace{1cm} (5)

where \( C \) is a list of \( 9^2 = 81 \) material parameters (Jaeger [1969]). The full stiffness tensor appears rather daunting at first glance. Thankfully, the symmetry of the stress and strain tensors immediately reduces this number to 36 independent terms. In order to prevent discontinuities in the strain
tensor, some additional degrees of freedom must be sacrificed. Formally stated, this is the requirement that a continuous medium deform continuously, and it reduces the number of independent material coefficients to 21. To gain further traction on higher dimensional problems, however, it is helpful to make a few additional assumptions.

2.3 General Assumptions

All of continuum mechanics rests upon the hypothesis that matter is infinitely divisible. While demonstrably untrue, this assumption generally yields accurate predictions down to the micron scale, and with a few adjustments the theory has been successfully applied even to the nanometer scale (Slattery and Fu [2004]). In most cases, we further assume that materials are homogeneous. In practice this means that we select a bulk material parameter that accounts for small variations within a material and that we pay close attention to the boundaries between one material and another, as in a composite beam. Sea ice, for example, retains brine pockets even in multi-year flows, but for bulk mechanical analysis it is standard practice to assign one value to the elastic modulus based upon overall density and salinity.

Computation becomes much simpler when material properties take on a degree of symmetry. If internal forces are carried and transmitted equally well in all directions, we say that the material is isotropic. In this case the number of independent material coefficients is reduced from 21 to just 2,
the elastic modulus $E$, and the Poisson ratio $\nu$. In general the shape of the
yield surface for sea ice is anisotropic (Taylor and Hatton [2006]), but for
the in-situ stress regime being investigated in this thesis, ice can be modeled
as isotropic without any loss of accuracy.

The use of any version of Hooke’s Law requires an assumption of elastic-
tility. Conceptually, elasticity is the idea that all strains due to loading are
immediately reversed when the load is removed. Mathematically, it means
that the coefficients relating stress to strain are independent of both stress
and strain. As loads increase, virtually all materials cease to be elastic be-
fore failure, but elasticity is often a valid assumption for materials well under
their yield stresses.

Putting these assumptions together we can write a generalized Hooke’s
Law which will be the basis for the consitutive equation used in this thesis:

$$
\begin{align*}
\epsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) \\
\epsilon_{xy} &= \frac{2(1 + \nu)}{E} \cdot \sigma_{xy} \\
\epsilon_{yy} &= \frac{1}{E} (\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) \\
\epsilon_{yz} &= \frac{2(1 + \nu)}{E} \cdot \sigma_{yz} \\
\epsilon_{zz} &= \frac{1}{E} (\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) \\
\epsilon_{xz} &= \frac{2(1 + \nu)}{E} \cdot \sigma_{xz}
\end{align*}
$$

(6)

The optically measured displacements in the calculations of this thesis
are made sense of as changes in $z$ position relative to a reference surface.
In this model, stresses are expected to accumulate as the underlying tidal
pressure increases and deforms the ice above it. The reference surface is a
smoothed version of the scan performed at low tide.
2.4 Thin Plate Small Deflection Theory

Plates are initially flat structures with one dimension that is considerably smaller than and relatively constant along the other two planar dimensions. The distinction between a thick plate and a thin plate is, unsurprisingly, a little murky, although most authors agree that a thickness to width ratio of 1/20 is sufficient to neglect any higher-order corrections to the assumptions that follow.

Figure 12 below depicts an exaggerated deformation of a section of thin plate. Note that the plane section that was initially normal to the dotted mid-surface remains planar and normal to the mid-surface after bending. This observation is a result of the fact that thin plates tend to deform in pure bending, and it is equivalent to stating that $\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} \approx 0$. In these circumstances, loads applied normal to the mid-surface are effectively transmitted as lateral stresses, and, provided there are no highly concentrated transverse loads, we may assume that $\sigma_{zz}=0$ as well.

Two additional assumptions can be made when the deformations (namely, $u_z$) are small in comparison to the plate thickness. First, any second order terms in the slope of the deformation are very close to zero, and second, we can assume that the mid-surface remains strain-free after bending. That is, there is no stretching of the middle plane of the plate. These assumptions are collectively known as the Kirchhoff hypotheses. A more detailed treatment of this material may be found in Ugural [1986].
Figure 12: An initially flat section of thin plate bends under an external load. The midsurface at $x = 0$ is displaced by an amount $u_z$.

With all of these additional assumptions, the compatibility equations for a thin plate undergoing small deformations can be expressed entirely in terms of the $z$-position, measured from the neutral mid-surface, and the spatial derivatives of the $z$-displacement function:

\[
\begin{align*}
\epsilon_{xx} &= -z \frac{\partial^2 u_z}{\partial x^2} \\
\epsilon_{yy} &= -z \frac{\partial^2 u_z}{\partial y^2} \\
\epsilon_{xy} &= -z \frac{\partial}{\partial x} \frac{\partial u_z}{\partial y}
\end{align*}
\]  

(7)
Note the simple $z$-dependence of strain. Because there are no vertical shears, strains will be maximized on either surface of the plate. A support in equilibrium with a thin plate will be subject to the stresses associated with these maximal strain values.

The generalized Hooke’s Law in Equation 6 is simplified considerably when the $z$-strain components are all zero. Here the constitutive law for a thin plate is expressed with $\sigma$ on the left for convenience in solving inverse problems.

\[
\begin{align*}
\sigma_{xx} &= \frac{E}{1 - \nu^2} (\epsilon_{xx} + \nu \epsilon_{yy}) \\
\sigma_{yy} &= \frac{E}{1 - \nu^2} (\epsilon_{yy} + \nu \epsilon_{xx}) \\
\sigma_{xy} &= \frac{E}{1 + \nu} \cdot \epsilon_{xy}
\end{align*}
\] (8)

Just as in the simple one-dimensional case, it is possible to back-calculate the internal forces that caused a given deformation by applying compatibility equations followed by the relevant constitutive law.

2.5 Thin Plate Large Deflection Theory

When the displacement $u_z(x, y)$ is no longer small in comparison to the plate thickness, tensile stresses develop along the midplane as it is stretched into a new shape. A set of evenly spaced grid points on the surface of a plate undergoing large deflection does not remain evenly spaced, as in Figure 13 below.
Although the deflections are not assumed to be small compared to the thickness of the plate, they are still assumed small relative to the two planar dimensions. By applying the principle of minimum potential energy to a deformation involving stretching of the midplane, it can be shown that the relevant compatibility equation is:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial u_z}{\partial x} \right)^2 \\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y} + \frac{1}{2} \left( \frac{\partial u_z}{\partial y} \right)^2 \\
\varepsilon_{xy} &= \frac{1}{2} \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} + \frac{\partial u_z}{\partial x} \frac{\partial u_z}{\partial y} \right) 
\end{align*}
\]

Unlike with the small deformation theory, there are now \( u_x \) and \( u_y \) terms that we cannot express purely in terms of \( z \)-displacements. These stretch terms represent the in-plane strains pictured in Figure 13. Additionally, the strains are no longer independent of one another. In general, to solve an inverse problem for a plate undergoing large deformations, one would
need to measure these stretch terms directly by affixing a mesh grid of the appropriate size to the undeformed shape. For a structural plate with dimensions in the kilometer range, this technique becomes impractical. An alternative method would be to combine the above strain relations into a differential equation that can be propagated from a stationary boundary using only displacements in the $z$-direction.

The version of Hooke’s Law (Equation 8) printed in the thin-plate section is equally valid when deflections are not small compared to plate thickness. The difference between the theories for small and large deflection is in how the displacement of material is related to the components of the strain tensor. In either case, an inverse problem is solved by making measurements of the displacement function $u_z$ across a discrete field of points, then applying finite-element analysis to determine a set of strain components for each point. Once these components are known, the thin-plate constitutive law can be used to calculate the components of the stress tensor. Because the physically relevant parameters are the invariants of this tensor, it is then necessary to solve an eigenvalue problem at each point. The resulting map of peak normal and shear stresses across the surface is useful for predicting the loads that might be transmitted to a structure in equilibrium with the plate.
2.6 The Mathematics of the Stretch Terms

Since the strains in equation 9 are not independent, it is possible to derive a relation between them. By differentiating Equation 9-a twice with respect to \( y \), Equation 9-b twice with respect to \( x \), and Equation 9-c by \( x \) and \( y \), one can show that

\[
\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \left( \frac{\partial^2 u_z}{\partial x \partial y} \right)^2 - \frac{\partial^2 u_z \partial^2 u_z}{\partial x^2 \partial y^2} \tag{10}
\]

Rewriting the left side of the equation in terms of displacement derivatives yields:

\[
\frac{\partial^2}{\partial y^2} \left( \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial u_z}{\partial x} \right)^2 \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial u_y}{\partial y} + \frac{1}{2} \left( \frac{\partial u_z}{\partial y} \right)^2 \right) - 2 \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial x} \frac{\partial u_z}{\partial y} \right) = \\
- \frac{\partial^2}{\partial y^2} \frac{\partial u_x}{\partial x} - \frac{\partial^2}{\partial x^2} \frac{\partial u_y}{\partial y} + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \frac{\partial u_z}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \frac{\partial u_z}{\partial y} \right)^2 - 2 \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial u_z}{\partial x} \frac{\partial u_z}{\partial y} \right) \tag{11}
\]

where the bolded terms are the needed quantities for \( \epsilon_{xx} \) and \( \epsilon_{yy} \) and all other terms can be calculated from the known function \( u_z(x, y) \). Finally, the differential equation in the most useful form becomes

\[
\frac{\partial^2}{\partial y^2} \frac{\partial u_x}{\partial x} + \frac{\partial^2}{\partial x^2} \frac{\partial u_y}{\partial y} = \\
\frac{\partial^2}{\partial y \partial x} (\nabla \cdot f) = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \frac{\partial u_z}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \frac{\partial u_z}{\partial y} \right)^2 \\
- 2 \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial u_z}{\partial x} \frac{\partial u_z}{\partial y} \right) + \left( \frac{\partial^2 u_z}{\partial x \partial y} \right)^2 - \frac{\partial^2 u_z \partial^2 u_z}{\partial x^2 \partial y^2} \tag{12}
\]
where $f$ is defined as $f_x = u_y; f_y = u_x$. Given the proper boundary conditions, this equation could at least in principle be solved over a discrete set of points by using finite difference methods to propagate a solution. However, in the case that the stretch terms are small and dwarfed by uncertainties in the model, they may be usefully approximated using a simplified geometry. In the following section I will derive a set of stretch terms whose influence on the final stress values can be evaluated along these lines.

2.7 Stretch Terms in 1 Dimension

![Figure 14: A simple deformation of a horizontal beam into the arc of a circle.](image)

Figure 14: A simple deformation of a horizontal beam into the arc of a circle.

According to basic beam-bending theory, the simplest deformation possible from a uniform load in one dimension is that of a circular arc. As a simple approximation for the stretch terms, I will assume that this is the shape taken at high tide. Using representative values for the distance
between the peninsula and the boundaries of the fjord and maximal tidal displacement, I will solve the stretch term for a one-dimensional geometry and then proceed as if these distances were radially symmetric.

If, in Figure 14, we take \( h \) to be 1m and \( c \) to be 1km, trigonometry gives a radius of curvature of 125,000m. Here the desired stretch term is \( \frac{du}{dx} \); the displacement function that tells how far each infinitesimal segment of horizontal line must move in the x direction to be transformed into the circular arc while maintaining uniform density. To find this function, we can divide the initial straight line into a series of infinitesimal strips of width \( dx \) and compare the amount of material in each strip of the domain before and after deformation.

Figure 15: The infinitesimal arc length is determined by the infinitesimal segment on the x domain.

According to Figure 15,
\[dl = \frac{dx}{\sin(\theta)}\]  

(13)

where \(\theta\) is the angle that the circle’s tangent makes with the vertical. This is equivalent to the typical polar coordinate, measured counterclockwise from the x axis. The difference between these two infinitesimals per unit length is given by

\[
\frac{1}{dx} (dl - dx) = \frac{1}{dx} \left( \frac{dx}{\sin(\theta)} - dx \right) = \frac{1}{\sin(\theta)} - 1
\]

(14)

This quantity is the amount of inward flux required to maintain constant density as the line deforms. But this linear flux is the spatial derivative of the displacement function, the same quantity that appears as the stretch term in the large deformation theory!

In this derivation I have assumed that the density of the undeformed chord is equal to the density of the deformed arc. This is not true, but the discrepancy between these two lengths goes as \(2r(\phi - \sin\phi)\), which can be shown to be a third order correction by taking a series expansion of the difference.

We can calculate the relevant domain in terms of \(\theta\) by referring back to Figure 14 and using trigonometry.
\[ \sin(\phi) = \frac{500m}{125000m} \]
\[ \phi = 0.00400 \] (15)

Thus the function \( \frac{du}{dx} \) on the domain \( 0m < x < 1000m \) is equivalent to the function \( \frac{1}{\sin(\theta)} - 1 \) on the domain \( \frac{\pi}{2} - 0.00400 < \theta < \frac{\pi}{2} + 0.00400 \). A graph of this estimated stretch term can be seen in Figure 16 below.

![Figure 16: Idealized stretch term for a 1-Dimensional geometry.](image)

Figure 16: Idealized stretch term for a 1-Dimensional geometry.
3 Experimental Methods

On March 23rd and March 24th, 2012, data was taken on the ice of Sveasundet using a 3D-laser scanner, the RIEGL VZ-1000. A full tidal half-cycle, between low tide at 9:49 am and high tide at 2:48 pm, was recorded in the form of eleven scans conducted from a stationary position on the peninsula. Three scans were also taken from the ice sheet itself, nearer the southern bank of the fjord. On March 24th, a pressure sensor was inserted into one of the minor cracks in the ice sheet and left for several tidal cycles.

Figure 17: A map of the acquired scan data in Sveasundet, with the pressure sensor indicated by white dot. Scans were performed from the cabin (scan position 1) at the end of the breakwater and from directly on the ice sheet (scan position 2). The coordinate system was imposed so that the y axis would be parallel to the thrust of the peninsula.
3.1 Running the Scanner

The VZ-1000 is a combination tachymeter and terrestrial laser scanner that uses pulses of infrared light to acquire position data for nearby objects and surfaces. A single $360^\circ$ scan produces a point cloud with tens of millions of members representing objects up to one kilometer away. The data is stored as ordered triplets in a coordinate system in which the scanner is located at the origin, and data can be viewed and manipulated in RIEGL’s accompanying software, RiScan.

On March 23rd, the VZ-1000 was mounted on the roof of a small cabin on the Fabrice breakwater. From this position scans were taken every half hour from low to high tide. See Table 1 for the scanning schedule. On March 24th, three scans were performed from the surface of the ice. These scans occurred during the first half of the rising tide cycle.
### Table 1: Laser Measurement Schedule

<table>
<thead>
<tr>
<th>March 23 (Cabin)</th>
<th>March 24 (Ice)</th>
</tr>
</thead>
<tbody>
<tr>
<td>hh mm ss</td>
<td>hh mm ss</td>
</tr>
<tr>
<td>09 49 20</td>
<td></td>
</tr>
<tr>
<td>10 22 37</td>
<td>10 41 34</td>
</tr>
<tr>
<td>10 49 12</td>
<td>11 27 20</td>
</tr>
<tr>
<td>11 21 38&lt;sup&gt;a&lt;/sup&gt;</td>
<td>12 14 08</td>
</tr>
<tr>
<td>11 50 06</td>
<td></td>
</tr>
<tr>
<td>12 20 07</td>
<td></td>
</tr>
<tr>
<td>12 50 26</td>
<td></td>
</tr>
<tr>
<td>13 20 05</td>
<td></td>
</tr>
<tr>
<td>13 50 02</td>
<td></td>
</tr>
<tr>
<td>14 20 03</td>
<td></td>
</tr>
<tr>
<td>14 48 02</td>
<td></td>
</tr>
</tbody>
</table>

<sup>a</sup> Time of major crack closure.

### 3.2 Scanner Limitations

In my experience, the maximum range for acquiring data on snow-covered ice is closer to 200m. Both snow and ice are opaque to 1-\(\mu\)m laser light, so the surface that appears in the VZ-1000's data is that of the snow atop the ice sheet. Although snow depth was observed to vary by as much as 40cm over the Sveasundet region, this fact does not impact calculations as long as no significant change in snow-depth-differential (such as would occur with drifting snow) occurs during the scan cycle. All deformations are calculated from a reference surface which can effectively absorb an uneven distribution of snow.

However, because the scanner must function as the origin of its data cloud, difficulties arise when scanning from atop a deforming surface. The
scanner was observed to tilt during the course of a single scan from the ice, creating a discontinuity between points along the azimuth where the scanner started and completed its 360° sweep. Because of this effect, the data from the ice can only be made sense of by taking a cross section which is parallel to the tilt of the scanner during these scans.

Figure 19: The cracks are maximally open at low tide. By the 11:21am scan, all primary cracks have closed.
3.3 Manipulating the Data

The RiScan program was then used to bring all data into a single coordinate system. The major cracks within the breakwater hinge zone were marked and exported as a separate data set. The time of crack closure was observed to be the 11:21 am scan, as in Figure 19.

RiScan’s weighted average algorithms were used to reduce the size of the pointcloud to approximately 280 points per square meter, or one point per 6x6 cm square. This data was then exported into MATLAB where a coordinate system orthogonal to the peninsula was imposed and the mesh was further reduced to one point per square meter. All 3D MATLAB plots appearing in this thesis are of calculations done on a 300x300m centered on the Fabrice cabin. In Figures 20 and 21, the same image of the Fabrice breakwater with cracks highlighted is shown in RiScan and MATLAB.

Figure 20: A height-colored map of the Fabrice peninsula at low tide in RiScan. Cracks appear in red.
Figure 21: A height-colored map of the Fabrice peninsula at low tide in MATLAB. The major cracks, the position of the scanner, and the position of the pressure sensor are plotted as black, red, and yellow stars, respectively.

Once imported into MATLAB, the position data for all eleven scans on the 300x300m grid was filled in using second-order interpolation in regions of scanner shadow. It is this final data set, a 300x300x11 matrix in which each value is a z position measured relative to the scanner, that was used to calculate stress and strain values.

3.4 Pressure Measurements

On March 24th a pressure sensor was placed within one of the minor cracks in the hinge zone beside the Fabrice breakwater. The sensor is capable of recording a one-dimensional stress value along its axis of orientation. The stress values recorded were those of $\sigma_{xx}$, the diagonal stress tensor component in the direction perpendicular to the thrust of the Fabrice breakwater.
While the method of placement the pressure sensor was crude, it will serve as the only direct force measurement made concurrently with the scans.

Figure 22: The location of the pressure sensor relative to the Fabrice cabin.
4 Results and Analysis

This section includes presentation and discussion of all results and calculations. The small deformation theory is observed to yield unreasonably high predicted stress values while the large deformation theory with simplified stretch term yields values between the range of measurements performed by Caline [2010] in years prior and those measured directly during the time of scanning. The median change in height for all datapoints recorded by the scanner from low to high tide was observed to be 1.12m, while the average ice thickness is estimated to be 1.3m. Generally a displacement to thickness ratio of $<0.2$ is desired for applying the small deformation theory (Ugural [1986]) to displacements, while a ratio of $>1$ is expected in invoking the large deformation theory. The encountered ratio of 0.86 demanded a treatment and comparison of results under both models.

4.1 Pressure Sensor Measurements

Figure 23 depicts the stress measurements made by the pressure sensor embedded within the ice. Internal forces are shown to be compressive with an average value of 330 kPa and a $<1\%$ fluctuation over the course of the tidal cycle. Due to the difficulty of the application, uncertainties of this method are thought to be as large as $+/ - 50\%$. 

40
4.2 Elastic Modulus

The primary material parameter used in elastic calculations is the elastic modulus, $E$. While the Poisson ratio is of equal weight mathematically, this number is generally not observed to vary from the value 0.3 for sea ice (Weeks [2010]). Temperature, density, and salinity profiles taken from Sveasundet cores in March 2012 were compared with similar measurements made in March 2007 in the same location. The fact that deviations between these measurements were small suggests that a published study of elastic modulus also conducted in March 2007 may be descriptive of the sea ice
encountered in this thesis. In Gabrielsen and Hoyland [2008], the modulus was observed to vary monotonically from 1.0 GPa at the ice foot to 1.5 GPa 15m from the coast of the breakwater. Level ice throughout Sveasundet was observed to have a modulus of 1.5 GPa.

For the calculations in this thesis, an elastic modulus of 1.5 GPa was used. Both small and large deformation theories yielded extremely high and physically nonsensical values for stress within the hinge zone, and this region, which extends as far 12m from the breakwater, has been cropped from the plots presented in the Analysis section. Additional plots, scaled to accomodate these large values appear in the Appendix.

4.3 Small Deformation Theory

In this sub-section, peak stress values are reported for the bottom of the ice sheet. Due to the geometric symmetry of the idealized plate shape, these values are equal and opposite to those on the top surface of the ice.

Figure 24 is a plot of stress values over time in a region defined by nine grid points in the vicinity of the pressure sensor. Typical values for calculated peak normal stress are on the order of tens of megapascal, while expected yield strength values for first-year sea ice are on the order of 1 MPa (J.A. and Jones [1993]). The reference surface has been taken as a median-smoothed image of the surface measured at low tide. Results are similar when the reference surface is taken to be the scan during which the major cracks close.
Figure 24: Stress versus time data for nine adjacent grid points at the location of the pressure sensor measurements. The temporal axis is given in scan number, equally spaced in time from low to high tide.

The calculated values for normal and shear stress are of similar magnitude throughout the larger Sveasundet region and cannot be interpreted in any physically meaningful way. Figures 25 and 26 show peak stresses calculated in each position over the length of the tidal cycle. In these plots, the region within the hinge zone cracks has been cropped for clarity.

Figure 27 depicts the trends in shear and normal stress over the duration of the tidal half-cycle. A distinction was made between the region of the ice that is near the scanner and that which is farther away. Short range shear stresses decrease by 50% while all other stress values appear to increase by...
Figure 25: A map of the highest calculated shear stress values in the Fabrice peninsula region over the course of a tidal cycle.

10-20%.
Figure 26: A map of the highest normal shear stress values in the Fabrice peninsula region over the course of a tidal cycle.
Figure 27: Median calculated normal and shear stress values in the Fabrice peninsula region over the course of a tidal cycle.
4.4 Large Deformation Theory

The normal stress values predicted by the large deformation theory at the location of the pressure sensor are shown in Figure 28. Median normal stress at high tide within this region is calculated to be 200 kPa.

Figure 28: Stress versus time data for nine adjacent grid points at the location of the direct pressure sensor measurements. The temporal axis is given in scan number, equally spaced in time from low to high tide.

Throughout the tidal cycle, no stresses in the megapascal range were calculated except in the immediate vicinity of the hinge zone cracks. See Figures 29 and 30. The fact that stresses quickly approach critical values at the edge of the cracks on all sides of the Fabrice peninsula suggests that the
The geometry of the cracked ice surface is one that could not occur for unbroken ice. As some smaller, secondary cracks were observed at the time of data acquisition, it is tempting to conclude that calculated stress values greater than the critical stress of sea ice are indicative of minor cracks that remain unseen below the snow surface.

Figure 29: A map of the highest calculated normal stress values in the Fabrice peninsula region over the course of a tidal cycle.

The shear forces are observed to be an order of magnitude lower than the normal forces for the entire Sveasundet region. However, they also abruptly approach critical stress values just outside the position of the major cracks.
Figure 30: A map of the highest calculated shear stress values in the Fabrice peninsula region over the course of a tidal cycle.
Figure 31 depicts the trends in median normal and shear stresses as the tide rises. Within 30m of the cabin, the effect of the tide is most clearly apparent. Here normal stresses increase by $\sim 100\%$ and shear stresses by $\sim 50\%$. Beyond 30m both the stresses and fractional increases are smaller.

Figure 31: A plot of the median calculated normal and shear stress values for regions at short (within 30m of scanner) and long (more than 30m from scanner) distances.

For the large deformation theory in general it can be said that forces in the coastal ice are in the range of a few hundred kilopascals while the forces in the ice further from the shore are only tens of kilopascals.
4.5 Discrepancies Between the Two Theories

The results of the small and large deformation theories are expected to agree at some intermediate value of displacement to thickness ratio. In understanding the discrepancy encountered in this data set, I have appealed to a figure in Ugural [1986] intended to help engineers estimate values for deflection and stress when the deformation is large but the linear small deformation theory has been applied.

![Graph showing discrepancies between small and large deflection theories](image)

**Figure 32:** Figure provided in Ugural [1986] for bridging the small and large deflection theories for a simply-clamped circular plate. The left graph is a plot of unitless deflection vs displacement to thickness ratio. The right plot shows the fractional discrepancies expected for a unitless stress value at the edges and center of the circular plate.

The left plot in Figure 32 shows the theoretically predicted unitless displacements for a given uniform load $p_0$ according to the small and large deflection theories. For a displacement to thickness ratio of 1, the same stress corresponds to a displacement roughly twice as large when applying
the large deformation theory instead of the small deformation theory. In other words, when solving an inverse problem as is done in this thesis, for a displacement to thickness ratio of 1, the large deformation theory is expected to yield values that are half those of the small deflection theory. This explains the direction of the discrepancy but not the magnitude encountered in this thesis.

It may be that the approximation of a simply-clamped circular plate is not an accurate reflection of the ice sheet encountered in Sveasundet. The observed vertical displacements, which did not vary smoothly with distance from the breakwater, may impact the small deformation theory more than they do the large deformation theory. This figure does not fully answer the observed discrepancy. It is included to show that there is indeed a theoretical basis for larger stresses being predicted by the small deformation theory and for stresses near the coast to be larger than those further out in the fjord.

4.6 The Impact of the Stretch Terms

All results presented thus far under the large deformation theory have included stretch terms calculated from the simplified one-dimensional geometry. In invoking this simplification, I stated that the usefulness of the approximation would depend upon the relative contribution of the simplified stretch term to the calculated strain tensor values. If the stretch term were to be only a fraction of the quadratic displacement term, then the approximation of a one-dimensional geometry would be permissible under the
larger simplifications made in deriving the constitutive law and compatibility equation in the Theory section.

Figure 33: Contribution of stress terms within the region of the pressure sensor.

Figure 33 shows the stretch term contribution to the calculations made in Figure 28. Values shown are responsible for between 5% and 10% of the reported maximum normal stress. At farther distances from the cabin, the stretch term accounts for \( \sim 25\% \) of calculated normal stress. Compare Figures 31 and 34.
Figure 34: A plot of the median calculated normal and shear stress contributions of the stretch term for regions at short (within 30m of scanner) and long (more than 30m from scanner) distances from the scanner.
4.7 Morphological Observations

As a surveying instrument, the RIEGL VZ-1000 is of course also effective at making other kinds of sight-specific measurements. In this section I outline a few of the non-continuum mechanics results that have been useful in understanding the physics of the Sveasundet ice sheet.

Behavior of North and South Cracks: While all of the largest cracks were observed to close by the time of the 11:21 am scan, different behavior was observed generally between the cracks that were south of the peninsula and those that were north of it. The northerly cracks closed sooner and were open less at low tide than the southerly cracks. South is the direction of the ocean, suggesting that the force of gravity may be pulling the southerly ice away from the breakwater and the northerly ice towards the breakwater. Although no trending gradient was observed in this direction, it’s possible that one exists at a larger scale, perhaps obscured by the snow cover, as the ice sheet extended for several kilometers oceanward from the Sveasundet region.

One-Dimensional Slice: Figure 35 is a cross-section taken across the full width of the fjord. A large vertical gradient that is seen between 400m and 500m from the cabin within an hour of low tide is absent less than two hours later. The change in cross section is indicative of the ice surface “bottoming out” on the sea floor of the fjord at low tide. Although no cracks were visible from the snow surface in this region, the degree of strain associated with such a change in cross-section would certainly correspond
to stresses surpassing their critical values.

**Rotating Block:** In the hinge zone southeast from the cabin, an unusual pattern of crack closure was observed. While one crack perpendicular to the peninsula was seen to close by the time of the 11:21 am scan, another crack a few meters away opened at this time and continued to increase in size until high tide. An irregular block traced out by these two cracks is shown in Figure 36.

A clearer picture of the hinge zone emerges when the position of the block is observed over a tidal cycle. Figure 37 is a height colored timeseries of the block at low and high tide. At low tide there exists a vertical gradient across the block. The edge which is nearer the peninsula is at a greater height than
Figure 36: The dimensions of an irregular block of sea ice, bordered by cracks that alternatively open and close with the rising tide.

the far edge, as is consistent with the picture of an ice sheet “hanging” from the landfast coastal ice. By high tide, this gradient has disappeared. It can be seen that both the block and the ice farther westward have swung up to the same height as the rubble ice directly beside the cabin. Complicated geometry of this sort is a clear violation of the assumptions required for simple plate-beding theory and helps to explain the large stres values derived within the hinge zone.
Figure 37: A timeseries of the irregular block from low to high tide in the RiScan program. Both images have been colored by height with the same scale. Warmer colors indicate increasing height.
5 Conclusion

In order to evaluate the effectiveness of the RIEGL VZ-1000 laser scanner for estimating internal forces of coastal sea ice, data was taken simultaneously in the form of optical measurements and with a traditional pressure sensor embedded in the ice. For the sea ice within 30m of the Fabrice cabin, stresses were predicted to increase with rising tide, reaching a median peak value of 200 kPa. Both this estimate and the results of the pressure sensor (330 kPa) fall within the realm of previous coastal ice stress measurements. See Table 2.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Location</th>
<th>Peak Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frederking et al [1986]</td>
<td>Adams Island</td>
<td>70 kPa</td>
</tr>
<tr>
<td>Sayed[1988]</td>
<td>Adams Island</td>
<td>350 kPa</td>
</tr>
<tr>
<td>Moslet[2001]</td>
<td>Svea</td>
<td>25 kPa</td>
</tr>
<tr>
<td>Nikitin et al [1992]</td>
<td>Okhotsk Sea</td>
<td>500 kPa</td>
</tr>
<tr>
<td>Caline and Barrault[2008]</td>
<td>Svea</td>
<td>100-150 kPa</td>
</tr>
</tbody>
</table>

All calculations were based on the assumptions of simple elastic plate-bending theory. Due to the comparative ease of the laser scanning method, (one of the reasons the estimated error on the pressure sensor values reported in this thesis is so large is the fact that properly embedding a sensor in 1m thick ice is a painstaking process that can take days of refinements as a rosette freezes and thaws in place) the RIEGL VZ-1000 is enthusiastically endorsed as a tool for determining site-specific internal sea ice forces.
5.1 Further Lines of Inquiry

Many of the authors listed in Table 2 encountered variations over the tidal cycle with peak stresses occurring at low tide. One proposed mechanism is the development of thermal stresses as the underlying temperature reservoir alternatively floods and recedes from the landfast and free floating ice.

While not strictly elastic, the primacy of thermal effects wouldn’t necessarily discredit optical methods. In fact, the choice of a reference surface is a relatively small part of the overall theory of plate-bending. Similar peak stress magnitudes would be calculated if the reference surface were chosen as the scan at high tide, only they would then be seen to occur at low tide. Since the pressure sensor used for this thesis did not vary with tide, it is difficult to speculate further. Scans must be taken of a more carefully monitored coastal region to better evaluate the accuracy of the plate-bending theoretical model.

The displacements encountered were slightly smaller than what is generally desired in applying the large deformation theory. Because of this, it would be worthwhile to evaluate stresses at different depths of the ice. The small deformation theory predicts 0 stresses due to pure bending at the mid-surface, while the large deformation theory, in the limit of a large stretch term, predicts equal stresses throughout the thickness of the ice. Comparing scan data with pressure sensor measurements made at different ice depths would help clarify the physics of plate bending when the displacement to thickness ratio is between 0.2 and 1.
6 Appendices

6.1 MATLAB Code

Small Deformation Theory Script

1 %This is the primary script for the Small Deformation calculations
3 load('results.mat')
4 icethickness =1.3; %meters
5 %To look at the region where the pressure data was measured directly,
6 %crop a 300x300 matrix like this: results(yrange,
7 %xrange,X,X)
8 yrange=149:151;
9 xrange=116:118;
10 %Set the reference z value matrix for a median--
11 %filtered low tide scan
12 refz=results(:, :,1,1);
refz=medfilt2(refz, [7 7]);

% Calculate the second spatial derivatives by composing MATLAB's gradient

for i=1:11
    [results(:,2,i) results(:,3,i)] = gradient(results(:,1,i)-refz,1);
    [~,results(:,5,i)] = gradient(results(:,2,i),1);  % d^2/dxdy
    [~,results(:,2,i)] = gradient(results(:,2,i),1);  % d^2/dx^2
    [trash results(:,3,i)] = gradient(results(:,3,i),1);  % d^2/dy^2
end

% STRAIN TENSOR

% by taking the positive value of the ice thickness we're calculating the peak stress at the bottom of the sheet, but the model is symmetric
26 for i=1:11
27 results(:,:,4,i)=icethickness*results(:,:,2,i); %Exx
28 results(:,:,5,i)=icethickness*results(:,:,5,i); %Exy
29 results(:,:,6,i)=icethickness*results(:,:,3,i); %Eyy
30 end

32 %ELASTIC STRESS TENSOR in Pa with E=1.5*10^9 newtons/m^2, v=.3
33 for i=1:11
34 results(:,:,7,i)=1.5E9/(1-(0.33)^2)*(results(:,:,4,i)
35 +0.33*results(:,:,6,i)); %sigmaxx
36 results(:,:,8,i)=1.5E9*results(:,:,5,i); %sigmaxy
37 results(:,:,9,i)=1.5E9/(1-(0.33)^2)*(results(:,:,6,i)
38 +0.33*results(:,:,4,i)); %sigmxy
39 end

39 %EIGENVALUE CALCULATIONS
40 for i = 1:300
41 for j=1:300
42 for k=1:11

63
\begin{verbatim}
results(i,j,10,k) = max(eig([[results(i,j,7,k),
    results(i,j,8,k); results(i,j,8,k), results(i,j,9,k)]));
results(i,j,11,k) = 1/2 * (max(eig([[results(i,j,7,k),
    results(i,j,8,k); results(i,j,8,k), results(i,j,9,k)]]) - min(eig([[results(i,j,7,k),
    results(i,j,8,k); results(i,j,8,k), results(i,j,9,k)]])));
end
end
end
\end{verbatim}

results = results(4:end-3,4:end-3,:) ;
% discard NaN padding

\% results is a 300x300x9x11 matrix. The fourth dimension separates
\% different scans in time. The third dimension is divided into position matrix, spatial derivatives,
\% strain/stress tensor elements as follows:

% 1st level: z positions
% 2nd level: d^2/dx^2(Z)
54 % 3rd level: $d^2/dy^2(Z)$
55 % 4th level: $Exx$
56 % 5th level: $Eyy$
57 % 6th level: $sigmaxx$
58 % 7th level: $sigmaxy$
59 % 8th level: $sigmayy$
60 % 9th level: peak normal stress ($sigma1$)
61 % 10th level: peak shear stress ($sigma1-sigma2$)/2

62 sensorresults=results(yrange,xrange, :, :); %pick out
the sensor region

63 %Make a map of the peak stresses over the tidal cycle

64 shearmax=zeros(length(results(:,1,11,1)),length(results(:,1,11,1)));
65 normalmax=zeros(length(results(:,1,11,1)),length(results(:,1,11,1)));

66 for i=1:length(results(:,1,11,1))
   for j=1:length(results(:,1,11,1))
      shearmax(i,j)=max(max(results(i,j,11,:)));
   end
end
peaknormal(i, j) = \max(\max(\text{results}(i, j, 10, :)))

end

end

for i = 1:length(results(:, 1, 11, 1))
    for j = 1:length(results(1, :, 11, 1))
        shearmin(i, j) = \min(\min(\text{results}(i, j, 11, :)))
    end
end

for i = 1:length(results(:, 1, 11, 1))
    for j = 1:length(results(1, :, 11, 1))
        if shearmin(i, j)^2 > shearmax(i, j)^2
            peakshear(i, j) = shearmin(i, j)
        else
            peakshear(i, j) = shearmax(i, j)
        end
    end
end

peakshear = abs(peakshear);

figure; mesh(pencrop3(medfilt2(peaknormal, [5 5])));
    axis([0 300 0 300 0 9E7]); caxis([0 5e7]); title('
Peak Normal Stress (Small Deformation);  
\[ \sigma \text{(Pa)} \]
view([-20 76])

figure; mesh(pencrop3(medfilt2(peakshear,[5 5])));
axis([0 300 0 300 0 9E7]); caxis([0 5e7]); title('
Peak Shear Stress (Small Deformation)');

zlabel('
\sigma \text{(Pa)}'); view([-20 76])

\%
Make a map of trending shear and normal stresses
\%
Define filters

[xx yy]=meshgrid(1:length(results(:,1,1,1)));
shortrange=sqrt((xx-length(results(:,1,1,1))/2).^2+(yy-
length(results(:,1,1,1))/2).^2)<=30;
longrange=sqrt((xx-length(results(:,1,1,1))/2).^2+(yy-
length(results(:,1,1,1))/2).^2)>30;
times=1:11;

for i =1:11

slice=pencrop3(results(:,10,i)); %grab normal

stress in a 2d array for filtering
normaltrend(i,1)=nanmedian(slice(shortrange));

slice=pencrop3(results(:,11,i)); %grab shear

stress in a 2d array for filtering
sheartrend(i,1)=nanmedian(slice(shortrange));

slice=pencrop3(results(:,10,i)); %grab normal stress in a 2d array for filtering
normaltrend(i,2)=nanmedian(slice(longrange));

slice=pencrop3(results(:,11,i)); %grab shear stress in a 2d array for filtering
sheartrend(i,2)=nanmedian(slice(longrange));

end

figure; plot(times, normaltrend(:,1), times, normaltrend(:,2), times, sheartrend(:,1), times, sheartrend(:,2))

legend(['\sigma: short range (<30m)'), ['\sigma: long range (>30m)'), ['\tau: short range (<30m)'), ['\tau: long range (>30m)'), 'Location', 'NorthWest');
title('Median Stresses Over Time (Small Deformation)'); ylabel('\sigma (Pa)'); xlabel('Scan No.');

%Save an array of stress and strain values at the pressure sensor in a
%yrange X xrange X 9 X 11 matrix called sensorresults

sensorresults = results(yrange, xrange, :, :);

%With sensorresults, let's look at the points surrounding the sensor (3,3)

figure;

for i = length(yrange)
    for j = length(xrange)
        plotthis = zeros(11, 1);
        plotthis2 = zeros(11, 1);
        for k = 1:11
            plotthis(k) = sensorresults(i, j, 7, k); %
            plotthis2(k) = sensorresults(i, j, 10, k); %
        end
        subplot(3, 3, ((i - 1) + (j - 2) * 3))
        plot(plotthis); hold all
        plot(plotthis2); hold off
        axis([1 11 −3E7 4E7]);
ylabel(‘$\sigma$’)

xlabel(‘Scan No.’)

legend(‘$\sigma_{}\{xx\}$’, ‘$\sigma_{}\{\text{max}\}$’, ‘Location’, ’Best’)

end
Large Deformation Theory Script

1 \%This is the primary script for the Large Deformation calculations

2 scantimes = ['09:49'; '10:22'; '10:49'; '11:21'; '11:50';
               '12:20'; '12:50'; '13:20'; '13:50'; '14:20'; '
               14:48'];

3

4 load('results.mat') \%load the 4d array results

5 icethickness = 1.3; \%in meters

6 \%To look at the region where the pressure data was measured directly,

7 \%crop a 300x300 matrix like this: results(yrange, xrange,X,X)

8 yrange = 149:151;

9 xrange = 116:118;

10

11 \%Set the reference z value matrix for a median-
    filtered low tide scan

12 refz = results(:, :, 1, 1);

13 refz = medfilt2(refz, [7 7]);

14
15 %Calculate the spatial gradient. Remember we’re
taking dZ/dx and dZ/dy

16 %where Z is the value of the data minus some reference
z value

17 for i=1:11

18 [ results(:,2,i) results(:,3,i)] = gradient(results(:,1,i)−refz,1);

19 end

20

21 Stretchterms=stretchterms(1000,1,length(results(:,1,1)));

22 %STRAIN TENSOR calculated as E_{ij} = 1/2 * (du_i/dj+du_j/di) + stretchterm

23 for i=1:11

24 results(:,4,i)=0.5*results(:,2,i)^2 + Stretchterms(:,1,i); %Exx

25 results(:,5,i)=0.5*results(:,2,i).*results(:,3,i) + Stretchterms(:,2,i);%Exy

26 results(:,6,i)=0.5*results(:,3,i).^2 + Stretchterms(:,3,i); %Eyy

27 end
29 %ELASTIC STRESS TENSOR in Pa with $E=1.5*10^9$ newtons/m$^2$, $v=0.3$

30 for $i=1:11$

31 results(:,:,7,i) = $1.5E9/(1-(0.33)^2) \times (results(:,:,4,i) + 0.33 \times results(:,:,6,i))$; %sigmaxx

32 results(:,:,8,i) = $1.5E9 \times results(:,:,5,i)$; %sigmayy

33 results(:,:,9,i) = $1.5E9/(1-(0.33)^2) \times (results(:,:,6,i) + 0.33 \times results(:,:,4,i))$; %sigmayy

34 end

35

36 %EIGENVALUE CALCULATIONS

37 for $i = 1:length(results(:,:,1,1))$

38   for $j=1:length(results(1,:,1,1))$

39     for $k=1:11$

40       results(i,j,10,k) = max(eig([results(i,j,7,k), results(i,j,8,k); results(i,j,8,k), results(i,j,9,k)]));

41       results(i,j,11,k) = $1/2 \times (\max(eig([results(i,j,7,k), results(i,j,8,k); results(i,j,8,k), results(i,j,9,k)])) - \min(eig([results(i,j,7,k), results(i,j,8,k); results(i,j,8,k), results(i,j,9,k)])))$

73
results(i,j,8,k);
results(i,j,8,k),
results(i,j,9,k));
end
end
end

%this crop discards the extrapolated padding that was
used to aid gradient calculations
results=results(4:end-3,4:end-3,:,:);

%results is a 300x300x9x11 matrix. The fourth
dimension separates
%different scans in time (see scantimes array defined
at the start of this script.
%The third dimension is divided into position matrix,
spatial derivatives, strain/stress
%tensor elements, and eigenvalues of the stress tensor
as follows:
% 1st level: z positions
% 2nd level: dZ/dx
% 3rd level: dZ/dy
% 4th level: Exx
% 5th level: Exy
% 6th level: Eyy
% 7th level: sigmaxx
% 8th level: sigmaxy
% 9th level: sigmayy
% 10th level: peak normal stress (sigma1)
% 11th level: peak shear stress (sigma1−sigma2)/2

%Make a map of trending shear and normal stresses
%Define filters

[xx yy]=meshgrid(1:length(results(:,1,1,1)));
shortrange=sqrt((xx−length(results(:,1,1,1))/2).ˆ2+(yy−length(results(:,1,1,1))/2).ˆ2)<30;
longrange=sqrt((xx−length(results(:,1,1,1))/2).ˆ2+(yy−length(results(:,1,1,1))/2).ˆ2)>30;
times=1:11;
for i=1:11
    slice=pencrop3(results(:,i,10)); %grab normal stress in a 2d array for filtering
normaltrend(i,1)=nanmedian(slice(shortrange));
slice=pencrop3(results(:,11,i)); %grab shear stress in a 2d array for filtering
sheartrend(i,1)=nanmedian(slice(shortrange));

slice=pencrop3(results(:,10,i)); %grab normal stress in a 2d array for filtering
normaltrend(i,2)=nanmedian(slice(longrange));
slice=pencrop3(results(:,11,i)); %grab shear stress in a 2d array for filtering
sheartrend(i,2)=nanmedian(slice(longrange));
end

figure; plot(times, normaltrend(:,1), times,
           normaltrend(:,2), times, sheartrend(:,1), times,
           sheartrend(:,2))

legend(['\sigma: short range (<30m)', '\sigma: long range (>30m)', '\tau: short range (<30m)', '\tau: long range (>30m)', 'Location', 'NorthWest']);
title('Median Stresses Over Time (Large Deformation)'); ylabel('\sigma (Pa)'); xlabel('Scan No.')
%Make a map of the peak stresses over the tidal cycle

shearmax=zeros(length(results(:,1,11,1)),length(results(1,:,11,1)));

normalmax=zeros(length(results(:,1,11,1)),length(results(1,:,11,1)));

for i=1:length(results(:,1,11,1))
    for j=1:length(results(1,:,11,1))
        shearmax(i,j)=max(max(results(i,j,11,:)));
        peaknormal(i,j)=max(max(results(i,j,10,:)));
    end
end

for i=1:length(results(:,1,11,1))
    for j=1:length(results(1,:,11,1))
        shearmin(i,j)=min(min(results(i,j,11,:)));
    end
end

for i=1:length(results(:,1,11,1))
    for j=1:length(results(1,:,11,1))
        if shearmin(i,j)^2>shearmax(i,j)^2
            peakshear(i,j)=shearmin(i,j);
        else
            peakshear(i,j)=shearmax(i,j);
        end
    end
end
peakshear(i,j)=shearmax(i,j);
end
end
peakshear=abs(peakshear);
figure; mesh(pencrop3(medfilt2(peaknormal,[5 5])));
axis([0 300 0 300 0 1E6]); caxis([0 5e5]); title('Peak Normal Stress (Large Deformation)');
zlabel('
$\sigma$ (Pa)'); view([-20 76])
figure; mesh(pencrop3(medfilt2(peakshear,[5 5])));
axis([0 300 0 300 0 1E6]); caxis([0 5e5]); title('Peak Shear Stress (Large Deformation)');
zlabel('
$\sigma$ (Pa)'); view([-20 76])

%Save an array of stress and strain values at the pressure sensor in a
%yrange X xrange X 9 X 11 matrix called sensorresults
sensorresults=results(yrange,xrange,:,:);
% Within sensor results, let's look at the points surrounding the sensor (3,3) using the 11:21 reference figure;

for i=2:length(yrange)
    for j=length(xrange)
        plotthis=zeros(11,1);
        plotthis2=zeros(11,1);
        for k=1:11
            plotthis(k)=sensorresults(i,j,7,k); % Calculated stress in the direction of the pressure measurement
            plotthis2(k)=sensorresults(i,j,10,k); % Calculated max normal stress
        end
        subplot(3,3,((i-1)+(j-2)*3))
        plot(-1*plotthis); hold all
        plot(-1*plotthis2); hold off
        axis([1 11 -800E3 100E3]);
        ylabel('
\sigma')
        xlabel('Scan No.')
legend( {'\sigma_{xx}'}, {'\sigma_{\max}'},'
Location','Best')

title(['Position' strcat(int2str(i),',','
int2str(j))]);

end

end
Function to Calculate Simplified Stretch Terms

1 function Output = stretchterms(c, h, size)

2  %Script to calculate stretch terms from input chord length, height, and

3  %size of square matrix. Returns a 300x300x3 matrix of

4  u_x, u_x*u_y and u_y

5  Output=zeros(size, size, 3, 11);

6  for time=1:11

7      r = 1/(8*h*(time/11))*(c^2+4*h^2);  %Pythagorean

8          theorem to determine radius of curvature

9      phi1 = asin(0.5*c/r);  %calculate angular domain

10     domainphi = pi/2−phi1:2*phi1/1000:pi/2+phi1;

11     range = 1./sin(domainphi)−1;

12     domainr = 0:1:1000;

13     for i=1:400  %numerical integration of the stretch

14          term

15          intr(i)=sum(range(1:i));  end

16     for i=1:size  %convert radial vector field to

17          cartesian vector field

18          for j=1:size

19              r = sqrt((size/2−i)^2+(size/2−j)^2);

20              [val, ind] = min(abs(domainr−r));

21  end
\begin{verbatim}
17  Stretchr(i,j)=intr(ind);
18  angle=atan2(i-size/2,j-size/2);
19  Output(i,j,1,time)=Stretchr(i,j)*cos(angle);
20  Output(i,j,2,time)=Stretchr(i,j)*sin(angle);
21  end
22  end
23  [gradxx gradxy] = gradient(Output(:,:,1,time));
24  [gradyx gradyy] = gradient(Output(:,:,2,time));
25  Output(:,:,1,time)=gradxx;
26  Output(:,:,2,time)=gradxy.*gradyx;
27  Output(:,:,3,time)=gradyy;
28  end
\end{verbatim}
Function to Crop the Hinge Zone

1 function g = pencrop3(mat)
2
3 g=mat;
4 load('extrapolatedcracks.mat');
5 crack0949val=isnan(crack0949);
6 crack0949(crack0949val)=10;
7 crack0949(crack0949>9)=0;
8
9 for y=132:300
10     for x=find(crack0949(y,:),1):find(crack0949(y,:),1,'last')
11         g(y,x)=NaN;
12     end
13     end

15 end
6.2 Additional Figures

Figure 38: Peak normal stresses with hinge zone included.
Figure 39: Peak shear stresses with hinge zone included.

Figure 40: Trending stresses when hinge zone is included.
Figure 41: Peak normal stresses with hinge zone included.

Figure 42: Peak shear stresses with hinge zone included.
Figure 43: Trending stresses when hinge zone is included.
## List of Figures

1. Minimum extent of Arctic sea ice in 2012 with 30 year average highlighted. (NASA) .......................................................... 5

2. Temperature of maximum density, $T_{\text{max}}$, and freezing temperature, $T_f$, vs salinity (in ppt) for sea ice. The change in slope of the $T_f$ line at -8.2° occurs due to precipitation of $Na_2SO_410H_2O$. ...................................................... 8

3. Brine tubes at the interface between water and sea ice. ........ 9

4. The Svalbard archipelago with the site of the experimental setup highlighted. .............................................................. 11

5. Aerial view of the breakwater at Sveasundet, June 2007. (UNIS) 12

6. Engineers and scientists rely upon constitutive laws (Hooke’s Law and its variants) and geometric compatibility to connect internal forces with displacements. ................................. 13

7. A rod of cross-sectional area $A$ is loaded uniaxially in tension. 14

8. In the one-dimensional case, we can divide the material into intervals of equal length and calculate the strain in each. Geometric compatibility gives us a requirement for the corresponding displacement function. ................................. 16

9. One-dimensional compatibility equation and constitutive law. 17
10 Stress components acting on an infinitesimal cube. 18

11 The strain tensor components are related to changes in length and angle of a deforming infinitesimal cube. For clarity, in this figure and in Equation 4 below, only one non-zero off-diagonal component has been included. 19

12 An initially flat section of thin plate bends under an external load. The midsurface at $x = 0$ is displaced by an amount $u_z$. 24

13 A grid that is laid atop an initially flat surface must stretch to accommodate deformations. 26

14 A simple deformation of a horizontal beam into the arc of a circle. 29

15 The infinitesimal arc length is determined by the infinitesimal segment on the $x$ domain. 30

16 Idealized stretch term for a 1-Dimensional geometry. 32

17 A map of the acquired scan data in Sveasundet, with the pressure sensor indicated by white dot. Scans were performed from the cabin (scan position 1) at the end of the breakwater and from directly on the ice sheet (scan position 2). The coordinate system was imposed so that the $y$ axis would be parallel to the thrust of the peninsula. 33
18 The RIEGL VZ-1000 with specifications from the manufacturer’s datasheet. Throughput is the upper bound on point measurements per second.

19 The cracks are maximally open at low tide. By the 11:21am scan, all primary cracks have closed.

20 A height-colored map of the Fabrice peninsula at low tide in RiScan. Cracks appear in red.

21 A height-colored map of the Fabrice peninsula at low tide in MATLAB. The major cracks, the position of the scanner, and the position of the pressure sensor are plotted as black, red, and yellow stars, respectively.

22 The location of the pressure sensor relative to the Fabrice cabin.

23 Stress versus time data for March 24-25, 2012.

24 Stress versus time data for nine adjacent grid points at the location of the pressure sensor measurements. The temporal axis is given in scan number, equally spaced in time from low to high tide.

25 A map of the highest calculated shear stress values in the Fabrice peninsula region over the course of a tidal cycle.

26 A map of the highest normal shear stress values in the Fabrice peninsula region over the course of a tidal cycle.
27 Median calculated normal and shear stress values in the Fabrice peninsula region over the course of a tidal cycle. 46

28 Stress versus time data for nine adjacent grid points at the location of the direct pressure sensor measurements. The temporal axis is given in scan number, equally spaced in time from low to high tide. 47

29 A map of the highest calculated normal stress values in the Fabrice peninsula region over the course of a tidal cycle. 48

30 A map of the highest calculated shear stress values in the Fabrice peninsula region over the course of a tidal cycle. 49

31 A plot of the median calculated normal and shear stress values for regions at short (within 30m of scanner) and long (more than 30m from scanner) distances. 50

32 Figure provided in Ugural [1986] for bridging the small and large deflection theories for a simply-clamped circular plate. The left graph is a plot of unitless deflection vs displacement to thickness ratio. The right plot shows the fractional discrepancies expected for a unitless stress value at the edges and center of the circular plate. 51

33 Contribution of stress terms within the region of the pressure sensor. 53
A plot of the median calculated normal and shear stress contributions of the stretch term for regions at short (within 30m of scanner) and long (more than 30m from scanner) distances from the scanner. 54

A one-dimensional cross section of the Sveasundet ice surface as it appears in scans made from the cabin and from the ice. 56

The dimensions of an irregular block of sea ice, bordered by cracks that alternatively open and close with the rising tide. 57

A timeseries of the irregular block from low to high tide in the RiScan program. Both images have been colored by height with the same scale. Warmer colors indicate increasing height. 58

Peak normal stresses with hinge zone included. 84

Peak shear stresses with hinge zone included. 85

Trending stresses when hinge zone is included. 85

Peak normal stresses with hinge zone included. 86

Peak shear stresses with hinge zone included. 86

Trending stresses when hinge zone is included. 87
References


Bi Xiang-jun Yue Qian-jin, Yao Jian and Qu Yan. Ice induced fatigue of offshore structure and dynamic force research. *State Key Laboratory of Structural Analysis of Industrial Equipment, Dalian University of Technology*, 2004.