Pomona College

Senior Thesis in Physics

Non-commutative Geometry and Information Scrambling in Matrix Theory Black Holes

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Abstract

In recent years, there has been much controversy regarding the relationship between information and black holes. In order to avoid information loss, an acceptably general unitary model requires that infalling information be scrambled and then emitted as Hawking radiation. We set up a scheme to explore the influence of event horizon geometry on information scrambling for a black hole model from Matrix theory - M-theory in the light-cone frame - using highly-parallelizable Runge-Kutta evolution. Previous examination of this system considered only fermionic degrees of freedom. We include the coupling between fermionic and bosonic degrees of freedom. We find that information scrambling has a strange dependence on geometry – involving atypical periodic behavior – with no immediate interpretation.

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Chapter 1

Introduction – The Information Paradox

Two facts about black holes, when taken together, have unsettled physicists for decades. The first is that any form of "stuff," from photons to penguins, cannot escape after falling through a black hole's event horizon. The second is that over time, black holes evaporate out of existence via a process called Hawking radiation.[1] The first fact comes from general relativity, whereas the second comes from quantum theory.

Black holes seem to be pulling a fast one on us. We send some quantum state (our most general way of representing "stuff") through a black hole's event horizon, and inside the event horizon it must remain. After some finite time, the black hole has evaporated away and our quantum state is nowhere to be seen. This problem is called "the information paradox," since it is concerned with understanding what happens to the information contained in our state.[2]

[1] In this sense, a black hole is a *thermal* object: it has a temperature and an entropy associated with this process.

[2] One of the key lessons of quantum information theory is that information is physical [1]. It is worth noting how one might encode information in a quantum state. Compare, for instance, the classical byte 00101111 with the spin chain state $|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle$. The study of quantum information is richer that its

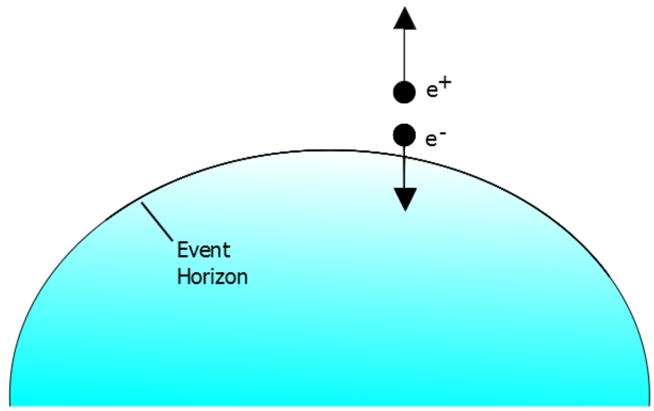


Figure 1.1: In Hawking radiation, particle-antiparticle (electron-positron in this figure) creation occurs just outside the black hole's event horizon in such a way that one particle (the electron) falls into the black hole while the other (the positron) escapes. This process causes black holes to evaporate over time.

A natural reaction to this thought-experiment is to suppose that our quantum state has been irretrievably lost. This idea is anathema to the modern physicist. In quantum mechanics, states are evolved in time via unitary transformations.[3] Unitary transformations are special because they preserve probability amplitudes.[4] Given an initial state $|\psi(0)\rangle$, we have that the state at time t is given by $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$, where $\hat{U}(t)$ is a unitary operator. If $\hat{U}(t)$ is unitary, we have that $\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\,\hat{U}^\dagger(t)\hat{U}(t)\,|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$; this forbids information loss.[5]

classical counterpart since in the quantum case one may consider information states that are superpositions of multiple states.

[3]This is a direct result of the Schrödinger equation. If you aren't famililar with unitary transformations, see section 1.4.

[4]In other words, for a unitary transformation U and any two states $|\psi\rangle$ and $|\phi\rangle$, we have that $\langle\phi|\,U^\dagger U\,|\psi\rangle = \langle\phi|\psi\rangle$. This is an equivalent definition to those given in section 1.4.

[5]Another way of seeing that unitary operators forbid information loss is by noting that that the inverse operator $U^\dagger$ is always well-defined. Via this inverse we may always reconstruct our initial state. However,

If $\hat{U}(t)$ were *not* unitary, however, we would have that $\langle\psi(t)|\psi(t)\rangle \neq 1$. Imagine preparing an electron in the spin state $|\psi(0)\rangle = |+z\rangle$, waiting for some time t, and then measuring the z-component of the electron's spin. Non-unitary evolution would allow the particle's final spin state to be $|\psi(t)\rangle = \frac{1}{\sqrt{3}}|+z\rangle + \frac{1}{\sqrt{3}}|-z\rangle$. According to the probabilistic interpretation of quantum mechanics, a subsequent measurement of the z-component of the particle's spin would have a $\frac{1}{3}$ chance of measuring $+\frac{\hbar}{2}$ and a $\frac{1}{3}$ chance of measuring $-\frac{\hbar}{2}$. What about the remaining $\frac{1}{3}$? We might try to interpret the missing $\frac{1}{3}$ as the probability that the final measurement returns no value because the electron has "disappeared." However, since such an operator $\hat{U}(t)$ is forbidden from being a time-evolution operator by the Schrödinger equation, we shouldn't worry too much about interpreting this strange final state.

The most popular proposed resolution to the information paradox that doesn't invoke information loss supposes that an in-falling quantum state is communicated to the outside world via the black hole's evaporation process. Most formulations of this idea invoke a concept called *Black Hole Complementarity.*

1.1 Black Hole Complementarity

Black Hole Complementary (BHC)[2] supposes a dual description of black holes. To the outside observer, the event horizon has microphysical degrees of freedom. If an outside observer decides to jump onto the horizon, however, she quickly discovers that it has no substance, and she falls through. The reason for BHC's name is that there is complementarity between observations made inside the event horizon and those made in the outside universe. Namely, the observer inside the horizon does not observe the horizon to have substance, but is unable to report this lack of substance to her colleagues in the outside universe due to the standard black hole gravitational limitations.

since the inverse is necessarily linear, we have that $U^\dagger 0 = 0$. In other words, the only way for information to be lost when acted on by a unitary operator is for the information to never have existed in the first place.

Many attempted resolutions to the information loss problem rely on BHC's interpretation of the event horizon as a physical membrane. To the outside observer, in-falling quantum states become entangled with the black hole's degrees of freedom by interaction with the event horizon membrane. The initial quantum state is then imprinted onto the black hole's Hawking radiation, and an outside observer is able to retrieve the information by collecting Hawking radiation.[6]

The BHC argument relies on three postulates, which assert the validity of (1) quantum theory, (2) semi-classical general relativity[7], and (3) statistical mechanics in describing black hole physics from the vantage point of a distant observer. An informal but important fourth postulate of BHC is that an in-falling observer "experiences nothing out of the ordinary" when crossing the event horizon. In other words, the in-falling observer detects no violations of the laws of physics. (For our case, the important law of physics that must be preserved is the No-Cloning Theorem of quantum mechanics.)

In 2012, a group from UCSB argued [3] that the postulates of BHC are inconsistent. The group proposed that a conservative resolution might be that in-falling observers burn up at the event horizon due to some sort of "firewall." Today, there is no consensus as to what happens when information crosses the event horizon. Although there has yet to be a resolution, these recent developments have caused many to reexamine quantum information in the context of black holes.

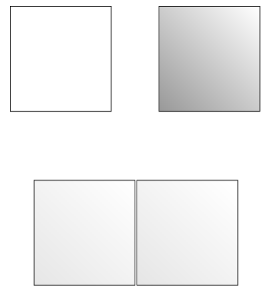


Figure 1.2: Objects at different temperatures equilibrate over time once they are put into thermal contact. This maximizes the total entropy of the system. We can imagine a quantum analogue to this process, in which two initially uncorrelated systems are allowed to interact (i.e. the system's Hamiltonian couples the subsystems). This interaction will cause the system's entanglement entropy to change.

1.2 Black Holes and the No-Cloning Theorem

We previously mentioned the idea that in-falling information becomes entangled with the black hole's degrees of freedom. This process is called *information scrambling*, and is most easily explained via a thermal analogy (see Fig. 1.2). Take two objects of different temperature and put them into thermal contact. After some time, the objects will have equilibrated to the same temperature. This maximizes the thermal entropy of the system, and thus is a *thermal* scrambling time. Analogously, take two quantum systems to be initially uncorrelated and let them interact. This interaction will cause the degree to which the subsystems are correlated with each other to change over time. In this case, the quantity we are concerned with is the system's *entanglement entropy*, the formula for which is given in Chapter 3. For some systems with initially uncorrelated subsystems, the entanglement entropy approaches

[6] BHC does not give the specifics of how in-falling information might interact with the membrane. It is not hard to believe, however, that via some mechanism some Hawking radiation photon could end up entangled with the spin state of an in-falling particle.

[7] Semi-classical general relativity is an approximation of quantum gravity in which matter is described via quantum mechanics and the spacetime metric (i.e. gravity) is treated classically.

some equilibrium value, around which it fluctuates.

We call a system *scrambled* if any subsystem with less than half of the system's degrees of freedom has some nontrivial fraction of the maximum possible entanglement entropy and exhibits thermal-like fluctuations in this entropy. Now take a scrambled system and add a degree of freedom. For our purposes, the initially scrambled system is the black hole and the new degree of freedom is some infalling qubit[8]. The new system will no longer be completely scrambled since adding a degree of freedom in a pure state will decrease the system's entanglement entropy significantly. After some time, the added information will diffuse over the black hole's degrees of freedom, and the new system will be scrambled. The amount of time this re-scrambling takes defines the *scrambling time* τ_{sc}.

For an infalling qubit, the black hole's scrambling time defines how long it takes for the entire black hole to "know" about the qubit. The idea is that subsequent Hawking radiation, assuming that it has some nontrivial interaction with the black hole, can also "learn" about the qubit from the black hole; a particle of Hawking radiation emitted after the scrambling time can then contain some or all of the information from the qubit.

It was shown by Hayden and Preskill [4] that, under the right conditions, the time it takes an observer outside the black hole to reconstruct the initial qubit by collecting Hawking radiation is approximately the scrambling time τ_{sc}. Thus, in the BHC picture, the information paradox is resolved. The firewall proposal tells us that BHC has some problems; in terms of the information paradox, there are still details to iron out. Information scrambling may still play an important role in resolving the information problem, though either in some tweak of the BHC paradigm or in some other theory altogether, and thus it is worth studying.

It turns out that we can put a lower bound on the scrambling time for a black hole. Consider the following thought experiment (Fig. 1.3). Alice and Bob are making observations from their spaceship just outside the event horizon of a black hole. Alice, who is holding a

[8]A qubit, or quantum bit, is a quantum state from a two-state (e.g. electron spin) system. When we have a series of qubits (e.g. several eletrons) interacting in some way, we call the system a qubit chain.

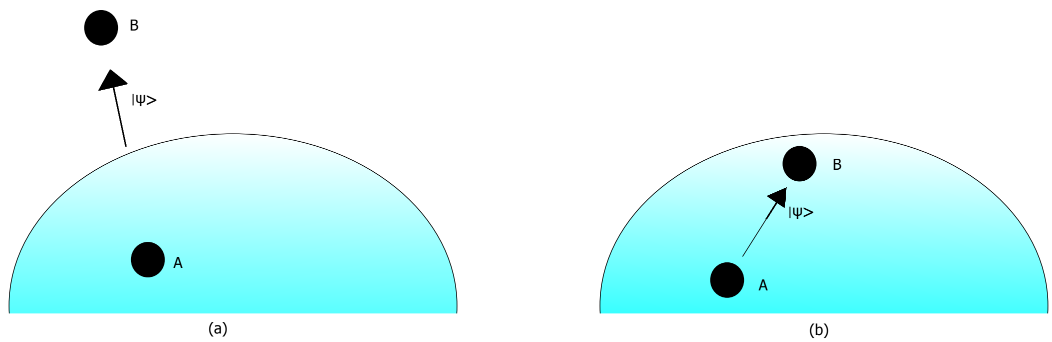


Figure 1.3: (a) Alice, holding the quantum state $|\psi\rangle$, jumps into the black hole. Bob waits outside, and after some time $\tau_{\rm sc}$ collects $|\psi\rangle$ from the Hawking radiation. (b) Bob then jumps into the black hole, and Alice sends him $|\psi\rangle$. Bob is only able to retrieve $|\psi\rangle$ before reaching the singularity if $\tau_{\rm sc} < \frac{\ln S}{T}$. This violates the no-cloning theorem.

qubit, decides to jump through the event horizon. Bob waits outside and collects the black hole's Hawking radiation. After some time, Bob has collected enough Hawking radiation to reconstruct Alice's qubit. Bob then jumps through the event horizon, where Alice has sent her qubit to Bob in the form of a photon. Then Bob has two identical copies of the same qubit, which violates the no-cloning theorem of quantum mechanics.

The way to resolve this problem is to put a lower bound on how long it takes Bob to collect Alice's qubit from the Hawking radiation. Then we may ensure that the qubit Alice carries into the black hole will not reach Bob before he meets his end at the singularity. In [5, 6], Sekino and Susskind argued that, to avoid a violation of the no-cloning theorem, the time it takes for Bob to collect Alice's qubit, and thus the scrambling time $\tau_{\rm sc}$ by the argument of Hayden and Preskill, is bounded by

$$\tau_{\rm sc} \geq \frac{\ln S}{T}, \tag{1.1}$$

where T is the black hole's temperature and S is its entropy. Most systems scramble as a power law in entropy by $\tau_{sc} \sim cS^{2/d}$, where c is a constant and d is the dimension of spacetime. Susskind and Sekino conjectured [5, 6] that black holes are *fast scramblers*, saturating the bound (1.1). This conjecture can be motivated by considering the rate of diffusion of charge on a black hole horizon or through Matrix theory, a framework for viewing M-theory.

1.3 Studying Scrambling via Matrix Theory

Matrix theory turns out to be an ideal setup for testing information scrambling in black holes. Since black holes are extremal gravitational objects and the information we are considering is quantum, it is reasonable to expect a quantum theory of gravity to be necessary in treating the dynamics of their information scrambling.[9] At this point, string theory is an attractive candidate for such a theory. Matrix theory [7, 8] is a particular formulation of string theory[10] that involves the dynamics of D0-branes coupled to each other by strings. D0-branes are the "point particles" of string theory.[11] Models for black holes in Matrix theory have been developed in [9, 10, 11] with promising results, giving the correct scaling for a Schwarzschild black hole's equation of state under certain assumptions. However, such models remain incomplete, and are unable to reproduce the correct entropy relations without additional dynamics.[12]

In this thesis, continuing on the work of [12], we set up a Matrix model for black holes to learn qualitative information about scrambling in quantum gravity. We put Matrix theory's bosonic variables – the spatial coordinates of and interactions between D0-branes – into a spherical configuration representing the black hole event horizon. We then allow for fermionic fluctuations, giving the system thermal behavior. Previous work considered only the coupling between fermionic degrees of freedom (ignoring the bosonic parameters that represent event horizon geometry) and found that the fermionic excitations had the structure of

[9] We should have suspected this, given that the information paradox arises from a conflict between general relativity and quantum theory.

[10] Specifically, M-theory is type IIA string theory in the strong coupling limit, and Matrix theory is M-theory viewed in the light-cone, or infinite-momentum, gauge.

[11] In general, Dp-branes are p-dimensional extended objects on which open strings end.

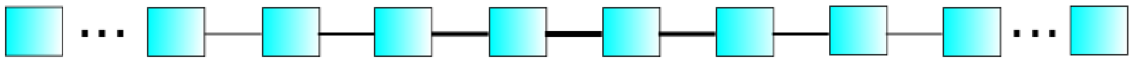


Figure 1.4: A qubit chain with nearest-neighbor coupling. Lines between boxes represent a couplings between qubits, with line thickness giving a sense of coupling strength. Intuitively, a nearest-neighbor coupling system ought to scramble more slowly than some qubit system with a denser network of interactions.

one-dimensional qubit chains with nearest-neighbor interactions (see Fig. 1.4). In this case, fast scrambling was not observed. To higher order, bosonic and fermionic degrees of freedom are in fact coupled, and the previous work qualitatively showed that these interactions were suggestive of fast scrambling.

In this thesis, we include the coupling between fermionic degrees of freedom (i.e. qubits living on the event horizon) and bosonic degrees of freedom (i.e. black hole geometry). We deform the event horizon from its initial spherical shape and analyze information scrambling among the qubits living on the deformed event horizon. The idea is that fluctuations in the event horizon occur naturally. Deforming the event horizon in extreme ways will help us understand the effect of smaller-scale fluctuations. We do this with a computational scheme that tracks entanglement over time for black hole systems with varying fermionic and bosonic parameters. We do not test the fast-scrambling conjecture for technical reasons; see Chapter 3. Our goal is to learn qualitative information about the effect of black hole geometry on information scrambling.

We found that information scrambling is highly dependent on deformation strength. For extreme deformations, we observe period behavior, strange for a system assumed to be psudo-thermal. While we have no immediate interpretation of our results, we suggest that the symmetries (or asymmetries) of our deformation modes, as well as the sizes of our deformations, play an important role.

In Chapter 2 we start from the Matrix theory Lagrangian and perform a fluctuation analysis on our black hole model to find a structure of qubit chains in the theory. In Chapter

3 we discuss our numerical techniques, which include highly-parallelized methods to explore very large Hilbert spaces. In Chapter 4 we present results. Chapter 5 gives conclusions and plans for future work. Technical details regarding gauge theory and spinors appear in Appendices A and B, respectively.

1.4 Notation, Some Important Mathematical Definitions, and Assumed Reader Knowledge

Throughout this thesis, repeated indices are summed over, except when explicitly stated:

$$x_i x_i + y_j y_j = \sum_i x_i x_i + \sum_j y_j y_j. \tag{1.2}$$

We do not deal with covariant and contravariant tensors, so we do not distinguish between lower and upper indices. The commutator of two matrices X and Y is given by

$$[X, Y] \equiv XY - YX. \tag{1.3}$$

The anticommutator is analogously defined by

$$\{X, Y\} \equiv XY + YX. \tag{1.4}$$

The Poisson bracket for two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ defined on N generalized position coordinates q_i, N generalized momenta p_i, and time t is given by

$$\{f, g\}_{\text{P.B.}} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \tag{1.5}$$

An important property of the Poisson bracket is that $\{q_i, p_j\}_{\text{P.B.}} = \delta_{ij}$.

A few types of matrices are especially important for our purposes. A matrix X is said to be *hermitian* if $X = X^\dagger$, where $X^\dagger$ denotes the conjugate transpose of X. A matrix U is called *unitary* if $U^\dagger U = UU^\dagger = I$. This is equivalent to requiring that $|\det U| = 1$. The group of all $N \times N$ unitary matrices is denoted $U(N)$.

The Kroenecker delta δ_{mn} equals 1 if $m = n$ and vanishes otherwise. The Levi-Civita symbol ϵ_{ijk} (also known as the antisymmetric symbol) is defined to vanish if any of the indices are equal, to be 1 for $i = 1, j = 2, k = 3$, and to pick up a negative sign for any permutation of those index values. (For example, $\epsilon_{123} = 1$, $\epsilon_{213} = -1$, and $\epsilon_{231} = 1$.)

Advanced undergraduate physics majors should be able to understand this thesis. In particular, a good grasp of undergraduate analytical mechanics (Lagrangian and Hamiltonian formulations) and quantum mechanics (Dirac notation, canonical quantization, density matrices) is assumed. Certain advanced topics (gauge theory, spinors) are developed in appendices.

Chapter 2

Matrix Theory

In this chapter, we develop a model for black holes in Matrix theory. We then analyze fluctuations in our fermionic degrees of freedom and find a series of coupled qubit chains. We will use these qubit chains to simulate information scrambling in a black hole.

2.1 The Lagrangian

The Lagrangian for Matrix Theory is given by [[12]]

$$L = \frac{1}{2}\mathrm{Tr}\left[(D_t X_i)(D_t X_i) + \frac{1}{2}[X_i, X_j][X_i, X_j] + \Psi D_t \Psi + \Psi \gamma_i [X_i, \Psi]\right], \tag{2.1}$$

which requires some explanation. The X_i's are $N \times N$ hermitian matrices, where i runs from 1 to 9. Ψ is also an $N \times N$ hermitian matrix, but its entries are ten-dimensional Majorana-Weyl spinors (See Appendix B). The γ_i are 16×16 Dirac matrices for 10-dimensional Minkowski space (see (B.2)) for our chosen representation). Our time dependence involves a "covariant time derivative" for the $U(N)$ gauge group

$$D_t \equiv \partial_t - i[A, \cdot], \tag{2.2}$$

where A is an $N \times N$ hermitian "gauge" matrix (see Appendix A). Since no time derivatives of A appear in the Lagrangian, A is static. In the end, since the Lagrangian needs to be a

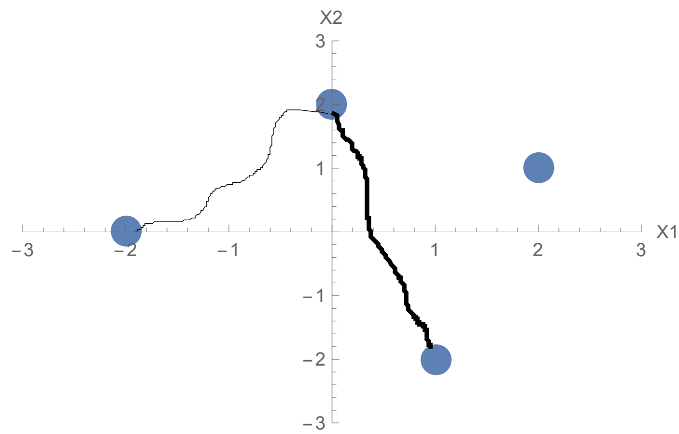


Figure 2.1: A plot of $N = 4$ Matrix theory with $X_1 = \begin{pmatrix} -2 & a & 0 & 0 \\ \bar{a} & 0 & b & 0 \\ 0 & \bar{b} & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 & c & 0 & 0 \\ \bar{c} & 2 & d & 0 \\ 0 & \bar{d} & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The thickness of the lines connecting different D0-branes represents the strength of their coupling, supposing that $|a| < |b|$ and $|c| < |d|$.

scalar, we calculate the trace of the bracketed quantity.

Physically, the X_i encode the spatial coordinates and interactions of N D0-branes (see Fig. 2.1). D0-branes are the point particles of string theory. The index i runs from 1 to 9 because Matrix theory has 9 spatial dimensions (with the "extra" 6 dimensions being curled up and thus imperceptible at low energies). Specifically, a diagonal element $X_i(n, n)$ is the i^{th} spatial coordinate of the n^{th} D0-brane, while an off-diagonal element $X_i(m, n)$ is the strength of the coupling in the i^{th} coordinate between the n^{th} and m^{th} D0-branes.

Since we want to model black holes using this theory, we fix a spherical configuration of D0-branes in the first three spatial dimensions, setting the other six coordinates of the D0-branes to zero. In other words, we ignore the "extra" dimensions of the theory. Allowing for deformations from the sphere, our ansatz for the X_i is

$$X_i = \nu\tau_i + x_i \quad i = 1, 2, 3, \tag{2.3}$$

where ν is a positive constant of our choice, the τ_i are $N \times N$ matrix representations of the generators of the algebra $\mathfrak{su}(2)$ and the x_i are $N \times N$ matrices. It is worth noting that the group $SU(2)$ corresponding to the algebra $\mathfrak{su}(2)$ is isomorphic to the sphere. The τ_i satisfy

$$[\tau_i, \tau_j] = 2i\epsilon_{ijk}\tau_k, \tag{2.4}$$

where ϵ_{ijk} is the Levi-Civita symbol. Thus the physically-minded reader may regard the τ_i as $N \times N$ generalizations of the Pauli matrices.[1] If $x_i = 0$ for $i = 1, 2, 3$, we have that

$$X_1^2 + X_2^2 + X_3^2 = \nu^2(\tau_1^2 + \tau_2^2 + \tau_3^3) = \nu^2\tau^2, \tag{2.5}$$

where $\tau^2 \equiv \tau_1^2 + \tau_2^2 + \tau_3^2$. Via standard quantum mechanics spin matrix constructions, one can show that

$$\tau^2 = \ell(\ell+1)I_N = \frac{N^2-1}{4}I_N, \tag{2.6}$$

with I_N the $N \times N$ identity matrix. We then have that

$$X_1^2 + X_2^2 + X_3^2 = \nu^2\frac{N^2-1}{4}I_N. \tag{2.7}$$

This equation is a noncommutative analogue (since matrices don't necessarily commute) of the equation for a sphere,

$$x^2 + y^2 + z^2 = r^2, \tag{2.8}$$

which suggests we should view this configuration as a sort of sphere with radius

$$R = \frac{\nu}{2}\sqrt{N^2-1}, \tag{2.9}$$

which goes to $\frac{\nu N}{2}$ in the large N limit. Of course, the X_i are matrices, so our geometry has noncommutative features. For that reason, this X_i configuration is called a "fuzzy" sphere.

[1] The algebra $\mathfrak{su}(2)$ can be represented by square matrices of any size. The Pauli matrices give rise to the 2-dimensional, or *fundamental*, representation of the $\mathfrak{su}(2)$ algebra. Here the algebra representation we use is determined by N, the number of D0-branes and the dimension of our matrices. We construct our τ_i in the same way one constructs higher-spin Pauli matrices in quantum mechanics. Then $N = 2\ell + 1$, where ℓ is the spin of the matrix representation.

The x_i are then our way of deforming our black hole's event horizon away from a fuzzy sphere configuration.

It is worth noting that this spherical configuration is unstable, although it has a long lifetime for large N. This decay process could provide a model for Hawking radiation, wherein black hole evaporation is represented by the D0-branes decoupling and moving out to infinity. We defer the analysis including the dynamics of the X_i to a later study. For our purposes, these bosonic degrees of freedom (i.e. the event horizon shape) will be static.

2.2 Matrix Decomposition

We next decompose our Ψ, A, and x_i matrices via spherical harmonics matrices Y^j_m:

$$\Psi_\alpha = \psi^j_{m\alpha} Y^j_m \quad , \quad A = a^j_m Y^j_m \quad , \quad x_i = x^j_{mi} Y^j_m, \tag{2.10}$$

where the spinor index α goes from 1 to 16 since Majorana-Weyl spinors in 10 dimensions have 16 components (see Appendix B). Since they come from the position matrices X_i, the x^j_{mi} are our bosonic degrees of freedom, while the $\psi^j_{m\alpha}$, arising from the spinor matrix Ψ (which represents fermions) are our fermionic degrees of freedom. We choose the gauge convention $a^j_m = 0$ (i.e. $A = 0$). The spherical harmonics matrices Y^j_m, where $j = 0, \ldots, N-1$ and $m = -j, \ldots, j$, are $N \times N$ matrices that form a basis for all $N \times N$ hermitian matrices. They are derived from the spherical harmonics of quantum mechanics in the next chapter. The Y^j_m satisfy

$$\mathrm{Tr}\left(Y^j_m Y^{j'}_{m'}\right) = (-1)^m N \delta_{jj'} \delta_{-mm'} \tag{2.11}$$

$$\left(Y^j_m\right)^\dagger = (-1)^m Y^j_{-m}. \tag{2.12}$$

Via the linearity of the matrix trace, we thus have that

$$\psi^j_{m\alpha} = \frac{(-1)^m}{N} \mathrm{Tr}\left(\Psi_\alpha Y^j_{-m}\right), \tag{2.13}$$

so, using the fact that the Ψ_α are hermitian (and thus $\Psi_\alpha = \Psi_\alpha^\dagger = (\psi^j_{m\alpha})^\dagger (Y^j_m)^\dagger = (-1)^m (\psi^j_{m\alpha})^\dagger Y^j_{-m}$), we find

$$
\begin{aligned}
(\psi^j_{m\alpha})^\dagger &= \frac{(-1)^{2m}}{N} \mathrm{Tr}\left(\Psi_\alpha Y^j_m\right) \\
&= (-1)^m \frac{(-1)^m}{N} \mathrm{Tr}\left(\Psi_\alpha Y^j_m\right) \\
&= (-1)^m \psi^j_{-m\alpha}.
\end{aligned} \tag{2.14}
$$

Via similar reasoning, one may show that

$$
(x^j_{mi})^\dagger = (-1)^m x^j_{-mi}. \tag{2.15}
$$

For the purposes of expanding and simplifying our Lagrangian, we ought to know how the Y^j_m matrices relate to our τ_i; then we may write the Lagrangian purely in terms of our expansion coefficients $\psi^j_{m\alpha}$ and x^j_{mi}. Thus, some important facts are that [?]

$$
\left[\tau_+, Y^j_m\right] = \sqrt{(j-m)(j+m+1)} Y^j_{m+1} \tag{2.16}
$$

$$
\left[\tau_-, Y^j_m\right] = \sqrt{(j+m)(j-m+1)} Y^j_{m-1} \tag{2.17}
$$

$$
\left[\tau_3, Y^j_m\right] = m Y^j_m, \tag{2.18}
$$

where $\tau_+ = \tau_1 + i\tau_2$ and $\tau_- = \tau_1 - i\tau_2$. Generally, we also have [?]

$$
\left[Y^j_m, Y^{j'}_{m'}\right] = f_{jmj'm'j''m''} (-1)^{m''} Y^{j''}_{m''}, \tag{2.19}
$$

where $f_{jmj'm'j''m''}$ is related to the Wigner $3j$ and $6j$ symbols[2] via

$$
\begin{aligned}
f_{jmj'm'j''m''} = &\frac{2}{N}(-1)^N N^{3/2} \sqrt{(2j+1)(2j'+1)(2j''+1)} \times \\
&\begin{pmatrix} j & j' & j'' \\ m & m' & m'' \end{pmatrix} \times \begin{Bmatrix} j & j' & j'' \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}
\end{aligned} \tag{2.20}
$$

[2]The Wigner $3j$ and $6j$ symbols (the round and curly-bracketed quantities in 2.20, respectively) are numbers related to the Clebsch-Gordan coefficients of quantum mechanics. For our purposes, it is sufficient to know that they can be looked up.

when $j + j' + j''$ is odd, otherwise evaluating to zero. Using these relations, we may write the Lagrangian in terms of the $\psi^j_{m\alpha}$, x^j_{mi}, and a^j_m.

The Hamiltonian splits as

$$H = H_x + H_{x,\psi} + H_\psi, \tag{2.21}$$

where H_x consists of only bosonic degrees of freedom, H_ψ consists of only fermionic degrees of freedom, and $H_{x,\psi}$ couples the two. Since we will be fixing the X_i, we can ignore H_x. The fermionic part of the Hamiltonian is, in term of ψ^j_m coefficients,

$$\begin{aligned} H_\psi = \frac{N}{4} \Bigg[& \sum_{j=0}^{N-1} \sum_{m=-j}^{j} (-1)^m \sqrt{1+j-m}\sqrt{j+m}\,\psi^j_{1-m}\gamma^1\psi^j_m \\ & + \sum_{j=0}^{N-1} \sum_{m=-j}^{j} (-1)^m \sqrt{1+j+m}\sqrt{j-m}\,\psi^j_{-1-m}\gamma^1\psi^j_m \\ & + i\sum_{j=0}^{N-1} \sum_{m=-j}^{j} (-1)^m \sqrt{1+j-m}\sqrt{j+m}\,\psi^j_{1-m}\gamma^2\psi^j_m \\ & - i\sum_{j=0}^{N-1} \sum_{m=-j}^{j} (-1)^m \sqrt{1+j+m}\sqrt{j-m}\,\psi^j_{-1-m}\gamma^2\psi^j_m \\ & - 2\sum_{j=0}^{N-1} \sum_{m=-j}^{j} (-1)^m m\,\psi^j_{-m}\gamma^3\psi^j_m \Bigg]. \end{aligned}$$

The fermionic-bosonic Hamiltonian is

$$H_{x,\psi} = \frac{1}{2} N f_{jm,j'm',j''m''} x^j_{mi} \psi^{j''}_{m''} \gamma^i \psi^{j'}_{m'}. \tag{2.22}$$

We now proceed to show that our fermionic degrees of freedom give rise to $8N^2$ qubits living in the black hole model.

2.3 Hilbert Space

In this section we find that our fermionic degrees of freedom satisfy certain special anti-commutation relations so that they may be interpreted as creation and annihilation operators for a collection of $8N^2$ fermions (or alternatively, a system of $8N^2$ qubits).

The canonical $\psi^j_{m\alpha}$ momentum is given by

$$\frac{\partial L}{\partial(\partial_t \psi^j_{m\alpha})} = \Pi^j_{m\alpha} = \frac{1}{2}(-1)^m N \psi^j_{-m\alpha} = \frac{N}{2}(\psi^j_{m\alpha})^\dagger. \tag{2.23}$$

Then classical mechanics gives us the Poisson bracket relation

$$\left\{\psi^j_{m\alpha}, \Pi^{j'}_{m'\alpha'}\right\}_{\text{P.B}} = \delta_{jj'}\delta_{mm'}\delta_{\alpha\alpha'}, \tag{2.24}$$

and thus

$$\left\{\psi^j_{m\alpha}, \left(\psi^{j'}_{m'\alpha'}\right)^\dagger\right\}_{\text{P.B}} = \frac{2}{N}\delta_{jj'}\delta_{mm'}\delta_{\alpha\alpha'}. \tag{2.25}$$

Via first canonical quantization, in which the Poisson bracket is replaced by an anticommutator[3] and the quantities inside the bracket are promoted to operators on some Hilbert space, we have

$$\left\{\psi^j_{m\alpha}, \left(\psi^{j'}_{m'\alpha'}\right)^\dagger\right\} = \frac{2}{N}\delta_{\alpha\alpha'}\delta_{jj'}\delta_{mm'}. \tag{2.26}$$

For $m > 0$, we rescale our fermionic variables by

$$s^j_{m\alpha} \equiv \sqrt{\frac{N}{2}}\psi^j_{m\alpha} \quad \text{for } m > 0, \tag{2.27}$$

giving us the anticommutation relations

$$\left\{s^j_{m\alpha}, (s^{j'}_{m'\alpha'})^\dagger\right\} = \delta_{\alpha\alpha'}\delta_{jj'}\delta_{mm'} \quad , \quad \left\{s^j_{m\alpha}, s^{j'}_{m'\alpha'}\right\} = 0 \quad , \quad \left\{(s^j_{m\alpha})^\dagger, (s^{j'}_{m'\alpha'})^\dagger\right\} = 0, \tag{2.28}$$

where $\alpha = 1, ..., 16$ and $m > 0$. This is the canonical creation/annihilation algebra for a collection of fermions. It is analogous to the algebra for the ladder operators of a harmonic oscillator, although in this case the use of anticommutators instead of commutators implies that our system is fermionic. Since $(\psi^j_{0\alpha})^\dagger = \psi^j_{0\alpha}$ by (2.14), the $m = 0$ case is special; if we define

$$\Gamma_{j\alpha} \equiv \sqrt{N}\psi^j_{0\alpha} \implies \{\Gamma_{j\alpha}, \Gamma_{j'\alpha'}\} = 2\delta_{\alpha\alpha'}\delta_{jj'}. \tag{2.29}$$

[3] Technically, we can have $\{f,g\}_{\text{P.B.}} \to -\frac{i}{\hbar}[f,g]$ or $\{f,g\}_{\text{P.B.}} \to -\frac{i}{\hbar}\{f,g\}$, with f and g being promoted to operators. Since the ψ^j_m come from a spinor, which represents fermions, it makes more sense to use the anticommutator. We also ignore the factor of $-\frac{i}{\hbar}$; it ends up not affecting our simulation.

This is the anticommutation relation for what we call a *Clifford algebra*. We will derive raising and lowering operators from the Clifford algebra in a moment.

For each $j = 1, \ldots, N-1$ and $m = 1, \ldots, j$, we have $16 \times j$ (from 16 possible α values and j possible m values) fermions that are created and annihilated with $(s^j_{m\alpha})^\dagger$ and $s^j_{m\alpha}$, respectively. Then we have $\sum_{j=1}^{N-1} \sum_{m=1}^{j} \sum_{\alpha=1}^{1} 16 = 8N(N-1)$ fermions from the $m > 0$ sector. The Clifford algebra, we will see, gives us an additional 8 qubits per j mode, for a total of $8N(N-1)+8N = 8N^2$ qubits. Thus our Hilbert space has dimension $2^{8N^2} = 256^{N^2}$.

We define our vacuum state $|\Omega\rangle$[4] by

$$s^j_{m\alpha} |\Omega\rangle = 0 \qquad j = 1, \ldots, N-1, \quad \alpha = 1, \ldots, 16 \tag{2.30}$$

and

$$\Gamma^-_{j\alpha} |\Omega\rangle = 0 \qquad j = 0, \ldots, N-1 \quad \alpha = 1, \ldots, 16 \tag{2.31}$$

where we have defined

$$\Gamma^+_{j\alpha} \equiv \frac{1}{2}(\Gamma_{j\alpha} + i\Gamma_{\alpha+8}) \quad , \quad \Gamma^-_{j\alpha} \equiv \frac{1}{2}(\Gamma_{j\alpha} - i\Gamma_{j\alpha+8}), \tag{2.32}$$

where $\alpha = 1, ..., 8$ and

$$\left\{\Gamma^+_{j\alpha}, \Gamma^-_{j'\alpha'}\right\} = \delta_{\alpha\alpha'}\delta_{jj'} \quad , \quad \left\{\Gamma^+_{j\alpha}, \Gamma^+_{j'\alpha'}\right\} = 0 \quad , \quad \left\{\Gamma^-_{j\alpha}, \Gamma^-_{j'\alpha'}\right\} = 0. \tag{2.33}$$

To get from a Clifford algebra to raising and lowering operators, one groups the initial operators (in our case, the $\Gamma_{j\alpha}$) into pairs. We have chosen to combine that α^{th} spinor component with the $\alpha + 8^{\text{th}}$ component. We chose this combination because, as we will soon see, it complements our chosen representation of the Dirac matrices. We may now generate any state in our Hilbert space by acting on $|\Omega\rangle$ with some number of creation operators. We may interpret our system as either creation/annihilation operators for a collection of

[4]It is interesting to show that a vacuum state exists. This is essentially because $\{s^j_{m\alpha}, s^j_{m\alpha}\} = 0$, implying that $(s^j_{m\alpha})^2 = 0$. For a full proof, see the spinor appendix.

fermions or as spin raising/lowering operators for a collection of qubits. In the fermion creation/annihilation interpretation, the raising operator with indices j, m, and α creates a fermion with those indices if such a fermion isn't already present; if such a fermion is already present, the state is killed since $\left((s^j_{m\alpha})^\dagger\right)^2 = (\Gamma^+_{j\alpha})^2 = 0$. The annihilation operator affects a state analogously, but removes the fermion with its indices. In the qubit interpretation, each raising operator turns a corresponding qubit on; applied more than once (or if the qubit is already in the "up" state), the state is killed. Thus we may interpret the 256^{N^2} states in the Hilbert space as all possible configurations of $8N^2$ qubits. We will use these qubits to simulate information scrambling on the fuzzy sphere.

2.4 The Fermionic Coupling Term H_ψ and Qubit Chain Structure

The fermionic Hamiltonian term, written in terms of raising and lowering operators and our chosen representation of the γ matrices (given in the Appendix), becomes

$$\begin{aligned} H_\psi = \nu N \sum_{j=0}^{N-1} \sum_{m=0}^{j} \sum_{\alpha=1}^{8} \Big[& 2m \left(\psi^j_{m\alpha}\right)^\dagger \psi^j_{m\alpha} - 2m \left(\psi^j_{m\beta}\right)^\dagger \psi^j_{m\beta} \\ & + i\sqrt{(j-m)(j+m+1)} (\psi^j_{m\alpha} \left(\psi^j_{m+1\beta}\right)^\dagger + \psi^j_{m\beta} \left(\psi^j_{m+1\alpha}\right)^\dagger \\ & + \left(\psi^j_{m\alpha}\right)^\dagger \psi^j_{m+1\beta} + \left(\psi^j_{m\beta}\right)^\dagger \psi^j_{m+1\alpha}) \Big], \end{aligned} \tag{2.34}$$

where $\beta \equiv \alpha + 8$. Note that the sum over α ranges from 1 to 8, with the qubit-qubit interactions coupling each αth qubit to the $\beta = \alpha + 8$th qubit. This is why we chose to get raising and lowering operators from our Clifford algebra by combining $\Gamma_{j\alpha}$ and $\Gamma_{j\alpha+8}$; the coupling between qubits has a clearer structure. We see that the qubits with different j don't interact. In fact, a qubit with indices j, α, and m only interacts with the qubits with indices j, β and $m \pm 1$. This nearest-neighbor structure is depicted in Fig. 2.2. We will see in the next chapter that when $H_{x,\psi}$ is also taken into account, the network of interactions

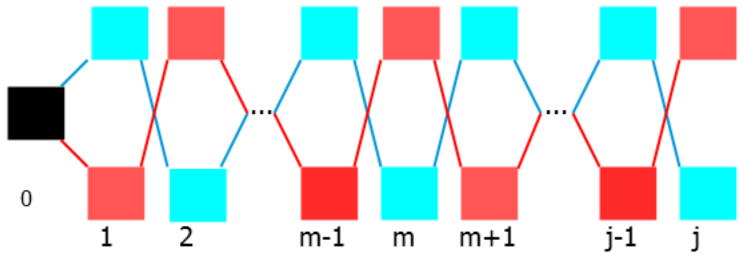


Figure 2.2: A graphical representation of the qubit-qubit coupling when only H_ψ is considered. Each square represents a qubit, with their m values listen below the chain. Each qubit with spinor index α is coupling to qubits with spinor index $\beta = \alpha + 8$. The $m = 0$ qubit from our Clifford algebra is drawn in black. We see that we have two chains, connected to each other through the $m = 0$ qubit.

between qubits is much more complicated. Moreover, since we are interested in the impact of black hole geometry on information scrambling, we will need to include the $H_{x,\psi}$ term in our simulation.

Chapter 3

The Simulation

In this chapter, we develop numerical techniques to analyze information scrambling in a qubit chain living on a Matrix theory black hole. Methods for calculating spherical harmonics matrices Y^j_m and deformation coefficients x^j_m are discussed. Certain computational concerns and simplifying approximations are also given.

3.1 Overview

The main idea of our simulation is to look at information scrambling in our qubit chain for various (fixed) deformations of the black hole's event horizon. Thus, when evolving the qubit chain in time, we need to make sure to include the Hamiltonian's fermionic-bosonic coupling term:

$$H_{x,\psi} = \frac{1}{2} N f_{jm,j'm',j''m''} x^j_{mi} \psi^{j''}_{m''} \gamma^i \psi^{j'}_{m'}, \tag{3.1}$$

where Einstein notation is being used. Since both x^j_{mi} and ψ^j_m appear, this part of the Hamiltonian couples the qubit chain to the black hole's geometry. Note that the ψ^j_m (without the lower index α) are spinors that are acted on by the γ matrices. Thus we can equivalently write $\psi^j_m \gamma_i \psi^{j'}_{m'}$ as $\psi^j_{m\alpha} \psi^{j'}_{m'\alpha'} \gamma^{\alpha\alpha'}_i$. At first glance, $H_{x,\psi}$ doesn't look so complicated. Upon further inspection, however, we see that the bosonic-fermionic interaction couples every qubit to every other qubit in the chain (with the strengths of the interactions dependent on the x^j_{mi} and the f coefficients). Also, whereas in the purely fermionic case only qubits with the same j are coupled, we now have coupling between qubits with different j. Moreover, in the purely fermionic case a qubit with index m only coupled to qubits with index $m \pm 1$, but when we take the bosonic-fermionic coupling into consideration all m values are coupled.

To simplify our computation, we will include only the bosonic-fermionic coupling for $i = 3$. In other words, we deform the event horizon from its initial spherical shape only in the X_3 direction, setting $x_1 = x_2 = 0$. We choose to deform in the X_3 direction because the form of our representation of γ^3 is especially simple (see Appendix). Moreover, we will fix the bosonic degrees of freedom (i.e. the black hole geometry will be static). This is because, as we will soon see, allowing for dynamics in the X_i would make our algorithm much more complicated. In other words, we will not include the H_x term in our Hamiltonian when evolving the system.

Via a straightforward but lengthy calculation, one may find the coupling term $H_{x,\psi}$ in terms of our raising and lowering operators and with $x_1 = x_2 = 0$. While this is necessary for running our simulation, its form is no more enlightening than (3.1). Moreover, the expanded coupling is exponentially more tedious to write out or look at. Thus we omit it here.

Since each α qubit is only coupled to other α qubits or $\beta = \alpha + 8$ qubits, we have that the $8N^2$ qubit system decouples into 8 qubit chains of length N^2. Looking at $H_{x,\psi}$, we see that all j and m modes are indeed coupled. Our network of interactions is thus much more complicated than the case when only H_ψ is considered. Also, the qubits are now coupled not only to each other, but to the black hole deformation parameters x^j_{m3}. Thus we expect more complicated dynamics when evolving the qubits in time.

Figure 3.1: Deforming the sphere. The spheroid on the right depicts a deformation in the Y_0^1 mode.

3.2 Evolving the Qubit Chain

The Schrödinger equation tells us that the time-evolution of some state $|\psi(t)\rangle$ is governed by the differential equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle . \tag{3.2}$$

For our purposes, $|\psi(t)\rangle$ is our qubit chain's state. The basis for our Hilbert space is the set

$$\{|\psi_1\rangle|\psi_2\rangle\ldots|\psi_{N^2}\rangle \ : \ |\psi_i\rangle = |\uparrow\rangle \text{ or } |\psi_i\rangle = |\downarrow\rangle \text{ for } i = 1,\ldots,N^2\}. \tag{3.3}$$

For example, if $N = 2$, then any state of the qubit chain can be written as a (normalized) linear combination of the basis states $|\downarrow\rangle|\downarrow\rangle$, $|\downarrow\rangle|\uparrow\rangle$, $|\uparrow\rangle|\downarrow\rangle$, and $|\uparrow\rangle|\uparrow\rangle$.

The initial state $|\psi(0)\rangle$ is chosen to be a random basis state. We calculate the x_{m3}^j as described in the Section 3.3 and load them into a text file that is read by the simulation. We then evolve the qubit chain in time using a highly-parallelized fourth-order Runge-Kutta method. The Runge-Kutta method is a standard tool used to numerically solve differential equations. For a good overview, see [13]. By highly-parallelized, we refer to additional numerical techniques that speed up our computation; see Section 3.5 for details.

3.3 Calculating the Spherical Harmonics Matrices Y^j_m and Deformation Constants x^j_{mi}

Recall that the spherical harmonic matrices Y^j_m played a key role in our expansion of the Matrix theory Lagrangian. We can also use the matrices to calculate the coefficients x^j_{mi} corresponding to a given deformation matrix x_i. We describe how to construct the matrices in this section, following [**?**], describe how to calculate the x^j_{mi} corresponding to some x_i, and give a few simplifying assumptions that allow us to easily explore the parameter space of deformations.

Let τ_i be defined as in (2.4), using the spin $s = \frac{N-1}{2}$ representation familiar from quantum mechanics. We then define the Y^j_m as $N \times N$ matrices that are polynomials of degree j in the τ_i, corresponding in some sense to the spherical harmonics functions from quantum mechanics, $Y^j_m(\theta, \phi)$. Let $j \in \{0, \dots, N-1\}$ and $m \in \{-j, \dots, j\}$ be given. Define the polynomial $y^j_m(r, \theta, \phi)$ by

$$y^j_m(r, \theta, \phi) \equiv r^j Y^j_m(\theta, \phi). \tag{3.4}$$

The $y^j_m(r, \theta, \phi)$ are homogenous polynomials of degree j in the variables $x \equiv r \sin\theta \cos\phi$, $y \equiv r \sin\theta \sin\phi$, and $z \equiv r \cos\theta$. In other words, in each term of $y^j_m(x, y, z)$, the sum of the powers of x, y and z is j. To get the Y^j_m out of $y^j_m(x, y, z)$, make the following substitution: write each term as $\frac{1}{j!}$ times the sum of each permutation of the elements in the term, ignoring commutativity for a moment. Then replace x with τ_1, y with τ_2, and z with τ_3. The resulting polynomial in τ_1, τ_2, and τ_3 defines Y^j_m. The reason for our strange substitution is that the matrices τ_i don't commute with each other, while the variables x, y, and z do. Thus there is some ambiguity in whether we write some term in $y^j_m(x, y, z)$ as xy or yx. This doesn't matter when x and y are commuting variables, but when we substitute them for matrices, the order matters. Thus the only "fair" way to write the term is $\frac{1}{2!}(xy + yx)$, since this gives us $\frac{1}{2!}(\tau_1\tau_2 + \tau_2\tau_1)$ every time. We will perform our simulations using $N = 4$, with our deformation matrices in the Y_0^j modes. For reference we list the $N = 4$ Y_0^j matrices here:

$$Y_0^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.5}$$

$$Y_0^1 = \begin{pmatrix} 1.34164 & 0 & 0 & 0 \\ 0 & 0.447214 & 0 & 0 \\ 0 & 0 & -0.447214 & 0 \\ 0 & 0 & 0 & -1.34164 \end{pmatrix}, \tag{3.6}$$

$$Y_0^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.7}$$

$$Y_0^3 = \begin{pmatrix} 0.447214 & 0 & 0 & 0 \\ 0 & -1.34164 & 0 & 0 \\ 0 & 0 & 1.34164 & 0 \\ 0 & 0 & 0 & -0.447214 \end{pmatrix}. \tag{3.8}$$

One reason that the spherical harmonics matrices Y_m^j are important is that they allow us to calculate the coeffecients x_{mi}^j corresponding to a particular deformation matrix x_i. Recall that $X_i = \nu\tau_i + x_i$. Our interpretation is that the X_i give the coordinates of N $D0$-branes and the strengths of their connections via strings. Thus we may control the black hole shape directly via the X_i. The fermionic-bosonic coupling $H_{x,\psi}$, however, which gives us the effect of the X_i on the evolution of the qubits, is expressed in terms of x_{mi}^j. Thus, if we want to deform the black hole in a certain way and see how information scrambling is affected, we need to know the x_{mi}^j corresponding to our desired deformation. To calculate x_{mi}^j for a particular bosonic configuration X_i, we need to use properties of the Y^j_m. Recall that

$$\mathrm{Tr}\left(Y^j_m Y^{j'}_{m'}\right) = (-1)^m N \delta_{-mm'}. \tag{3.9}$$

Therefore we have that

$$\mathrm{Tr}\left(X_i Y^j_{-m}\right) = (-1)^m N x^j_{mi}, \tag{3.10}$$

giving us a straightforward process, since we now know how to construct the Y^j_m, by which we may calculate the x^j_{mi}.

Moreover, the Lagrangian's residual gauge freedom allows for changes of basis[1], so we may diagonalize our X_3 matrix (since Hermitian matrices are always diagonalizable).[2] Since the undeformed X_3 is already diagonal, this means that we can restrict ourselves to considering diagonal deformation matrices. It turns out that each Y^j_0 is diagonal, so the set $\{Y^j_0 : j = 0, \ldots, N-1\}$ forms a basis for real diagonal $N \times N$ matrices. In other words, we only need to consider x^j_{03} for a diagonal deformation matrix x_3. This further simplifies our Hamiltonian, which is important in making our simulation computationally feasible. Choosing only $m = 0$ deformation modes also allows us to more easily explore the parameter space of deformations.

3.4 Quantifying Scrambling with Density Matrices

Our goal is to track information scrambling in the qubit chain as it evolves in time. To do this, we calculate the qubit chain's *entanglement entropy* at each time step. Entanglement entropy gives a measure of how entangled two subsystems are with each other. First we review density matrices, from which we may calculate entanglement entropy. For some state $|\psi\rangle$, the density operator is given by

$$\hat{\rho} = |\psi\rangle \langle\psi| \,. \tag{3.11}$$

[1]This is true even after we set $A = 0$; see Appendix A.

[2]Of course, this transformation will complicate X_1 and X_2 (since in general we cannot simultaneously diagonalize all three), but any change of basis won't change the fact that $x_1 = x_2 = 0$.

Recall that one may find the matrix representation of an operator given a particular basis by inserting the operator in the middle of an inner product of two basis states. In other words, the (i, j) entry of the matrix A corresponding to some operator $\hat{A}$ is given by

$$A_{ij} = \langle i| \hat{A} |j\rangle . \tag{3.12}$$

Density operators are more interesting for multiparticle systems than for single-particle systems. Consider the state $\psi = \frac{1}{\sqrt{2}} |\uparrow\rangle_1 |\uparrow\rangle_2 + \frac{1}{\sqrt{2}} |\downarrow\rangle_1 |\uparrow\rangle_2$ for a two-particle spin system, with each particle spin-$\frac{1}{2}$. Then our density operator is

$$\hat{\rho} = \frac{1}{2} (|\uparrow\rangle_1 |\uparrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2) (\langle\uparrow|_1 \langle\uparrow|_2 + \langle\downarrow|_1 \langle\uparrow|_2) . \tag{3.13}$$

The corresponding density matrix using the states $|\uparrow\rangle_1 |\uparrow\rangle_2 , |\uparrow\rangle_1 |\downarrow\rangle_2 , |\downarrow\rangle_1 |\uparrow\rangle_2$, and $|\downarrow\rangle_1 |\downarrow\rangle_2$ as a basis is

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.14}$$

What if we are only able to measure one of the particles? In that case, it is useful to define the *reduced* density operator. We define the reduced density operator for particle 1 by

$$\hat{\rho}_{(1)} = \sum_j \langle j|_2 \, \hat{\rho} \, |j\rangle_2 . \tag{3.15}$$

In other words, the reduced density matrix for particle 1 is the density operator traced over the basis for particle 2. For the full density operator in 3.13, we have the reduced density matrices

$$\rho_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} , \tag{3.16}$$

$$\rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.17}$$

The reduced density matrices ρ_1 and ρ_2 look quite different, but share important characteristics. Looking back at (3.13), we see that no matter which state we measure particle 1, particle 2 will have its spin pointing up. Moreover, since any measurement of particle 1's spin will give an answer of "up," we can't get any useful information about particle 2's spin by measuring particle 1. However, since $\rho_1^2 = \rho_1$ and $\rho_2^2 = \rho_2$, we have that both subsystems are in pure states. In other words, the system is in a product state. Since the definition of an entangled state is one that cannot be written as a product state, this means that the particles are minimally entangled with each other. As we will soon see, this means that their entanglement entropy is zero. For completeness, consider the density matrices for the entangled state $|\psi'\rangle = \frac{1}{\sqrt{3}} (|\uparrow\rangle |\uparrow\rangle + |\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\downarrow\rangle)$:

$$\rho' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}, \tag{3.18}$$

$$\rho_1' = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \tag{3.19}$$

$$\rho_2' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \tag{3.20}$$

In this case, ρ_1' and ρ_2' describe mixed states; the two particles are entangled with each other. We will see that these matrices have nonzero entanglement entropy.

Note that all definitions given so far are easily generalized to n-particle systems. From a reduced density matrix, we may calculate the *entanglement entropy.* For n-particle systems, we have more freedom in choosing which subsystem to trace over when calculating our reduced density matrix. Denote the subsystem of interest (i.e. the particles we don't wont to trace over) by $\mathcal{M}$. Then calculate the reduced density operator for $\mathcal{M}$:

$$\hat{\rho}(t)_{\mathcal{M}} = \mathrm{Tr}_{\overline{\mathcal{M}}} |\psi(t)\rangle \langle\psi(t)|, \tag{3.21}$$

where $\overline{\mathcal{M}}$ is the part of system not included in $\mathcal{M}$. As this is the computation we use in our simulation, we have explictly noted the time dependence of the operator. The entanglement entropy is then given by

$$S(t) = -\mathrm{Tr}\left[\rho(t)_{\mathcal{M}} \ln \rho(t)_{\mathcal{M}}\right], \tag{3.22}$$

where it is important to note that we are using the matrix logarithm[3], as opposed to simply applying the logarithm to each element of the matrix. The quantity in (3.22) is known as the *von Neumann entropy* of a density matrix. The entanglement entropy is simply the von Neumann entropy of a *reduced* density matrix. The von Neumann entropy quantifies how far a density matrix is from describing a pure state. Indeed, for a pure state,

$$\rho^2 = \rho, \tag{3.23}$$

which implies that $2\ln\rho = \ln\rho$. Therefore $\ln\rho = 0$, so the von Neumann entropy is zero. Moreover, the von Neumann entropy reaches its maximum for a maximally mixed state [1]. A maximally mixed state has density matrix $\frac{1}{N}I_N$, where N is the dimension of the Hilbert space and I_N is the N-dimensional identity matrix. Then the von Neumann entropy is $\ln N$. Thus the von Neumann entropy describes how mixed the system described by a density matrix is. If a reduced density matrix describes a pure state, we have that entire system is a product state. Recall that a system is called entangled if it cannot be written as a product state. Thus it makes sense for us to use the von Neumann entropy of a reduced density matrix as the entanglement entropy for the system.

Returning to our example calculations, the von Neumann entropies of ρ_1 and ρ_2 are zero, by the argument of the preceding paragraph. The fact that ρ_1' and ρ_2' describe mixed states suggest that their von Neumann entropies are nonzero. To calculate their entropies, we introduce a small trick. Matrix logarithms are, for general matrices, hard to compute. Note, however, that for a diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$, the matrix logarithm $\ln A =$

[3]For matrices, $\ln X = Y$ if $e^Y = 1 + Y + \frac{1}{2!}Y^2 + \frac{1}{3!}Y^3 + \cdots = X$.

$\text{diag}(\ln a_1, \ldots, \ln a_n)$. This is because, for a diagonal matrix $Y = \text{diag}(y_1, \ldots, y_n)$, we have that $Y^m = \text{diag}(y_1^m, \ldots, y_n^m)$. Therefore, if we have that $Y = \ln A$ for diagonal A, we have

$$\begin{aligned} e^Y = I + Y + \frac{1}{2!}Y^2 + \frac{1}{3!}Y^3 + \ldots = A \\ \text{diag}(1, \ldots, 1) + \text{diag}(y_1, \ldots, y_n) + \frac{1}{2!}\text{diag}(y_1^2, \ldots, y_n^2) + \ldots = \text{diag}(a_1, \ldots, a_n), \end{aligned} \tag{3.24}$$

giving us that $e^{y_i} = a_i$ for $i = 1, \ldots, n$. In other words, $y_i = \ln a_i$ for each i. Note that we were able to assume that Y is diagonal since Y^n is diagonal for every non-negative integer n, which is only true for diagonal matrices. Thus we have that the von Neumann entropy for a diagonal matrix $\rho_d = (\lambda_1, \ldots, \lambda_n)$ (i.e. ρ_d is written in its eigenbasis) is

$$\begin{aligned} S(\rho_d) &= -\text{Tr}\left[\rho_d \ln \rho_d\right] \\ &= -\text{Tr}\left[\text{diag}(\lambda_1, \ldots, \lambda_n)\text{diag}(\ln \lambda_1, \ldots, \ln \lambda_n)\right] \\ &= -\sum_{i=1}^{n} \lambda_i \ln \lambda_i, \end{aligned} \tag{3.25}$$

where we remember that $0 \ln 0 = 0$, which can be shown via a limit argument.

This is a cute computation, but only applies to diagonal matrices. The trick is that the von Neumann entropy is invariant under changes of basis, and thus this results applies for all diagonalizable matrices. To see the invariance of the von Neumann entropy, note that if $Y = \ln X$, we have

$$\begin{aligned} I + Y + \frac{1}{2!}Y^2 + \ldots = X \\ UIU^\dagger + UYU^\dagger + \frac{1}{2!}UY^2U^\dagger + \ldots = UXU^\dagger \\ I + UYU^\dagger + \frac{1}{2!}(UYU^\dagger)^2 + \ldots = UXU^\dagger, \end{aligned} \tag{3.26}$$

where $U^\dagger = U^{-1}$ and U are unitary matrices and we have used the fact that $UY^nU^\dagger = (UYU^\dagger)^n$ for all n. In other words, we have that $\ln(UXU^\dagger) = U \ln(X) U^\dagger$. Then we have that

$$
\begin{aligned}
S(UXU^\dagger) &= -\mathrm{Tr}\left[UXU^\dagger \ln(UXU^\dagger)\right] \\
&= -\mathrm{Tr}\left[UXU^\dagger U \ln(X) U^\dagger\right] \\
&= -\mathrm{Tr}\left[UX \ln(X) U^\dagger\right] \\
&= -\mathrm{Tr}\left[X \ln(X)\right], \qquad (3.27)
\end{aligned}
$$

where we have used the fact that the matrix trace is unchanged by a change of basis in the last step. Thus the von Neumann entropy of a density operator doesn't depend on which basis we write it in. Moreover, since density matrices are symmetric and non-negative by definition, they are always diagonalizable with real, non-negative eigenvalues. The for any density matrix ρ with eigenvalues $\lambda_1, \ldots, \lambda_n$, we have that

$$
S(\rho) = -\sum_{i=1}^{n} \lambda_i \ln \lambda_i. \qquad (3.28)
$$

This is the method we use in our simulation to calculate the entanglement entropy of $\rho(t)$. As an example, recall the reduced density matrices for our entangled two-particle system:

$$
\rho_1' = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \qquad (3.29)
$$

$$
\rho_2' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \qquad (3.30)
$$

A quick computation gives us that ρ_1' and ρ_2' have the same eigenvalues: 0.873 and 0.127. Therefore

$$
\begin{aligned}
S(\rho_1') = S(\rho_2') &= -\left[0.873 \ln(0.873) + 0.127 \ln(0.127)\right] \\
&= 0.380644, \qquad (3.31)
\end{aligned}
$$

the positive entanglement entropy that we expected for our entangled system. It is interesting that $S(\rho_1') = S(\rho_2')$. In fact, for a general partition $\mathcal{M}$ and $\overline{\mathcal{M}}$ of some system, one has that

$S(\rho_{\mathcal{M}}) = S(\rho_{\overline{\mathcal{M}}})$. We do not prove this fact here, but it further affirms that our definition of entanglement entropy is good. Entanglement is indeed a two-way street, and that should be reflected in our definition of entanglement entropy. For more on entanglement entropy, see [1].

3.5 Computational Challenges and Technical Details

The quantum gravity aspect of Matrix theory comes into play for $N \gg 1$. Recall that the dimension of our N^2-length qubit chain's Hilbert space is 2^{N^2}. Thus for just $N = 4$, the qubit chain's Hilbert space is $65,536$-dimensional. For $N = 4$, our simplified Hamiltonian has around $10,000$ terms. The number of terms in our Hamiltonian is a polynomial of order 6 in N. In other words, to get interesting results, we will need a very efficient algorithm. Thus we turn to parallelization. The basic idea of parallelization is to identify loops in a program's code with independent iterations and send different iterations of these loops to different GPUs. Without parallelization, our simulation would not be possible.

Alas, technical issues besides runtime also pop up. One issue with including the $H_{x,\psi}$ coupling is that our qubit chains become much larger. The fact that our qubit chain is of length N^2 means that it becomes impractical for us to test the fast scrambling conjecture, since at $N = 5$ and $N = 6$ the simulations begin to take prohibitively long and/or exceed our 12 Gigabytes of GPU memory. The previous work [12] was able to explore the fast scrambling conjecture precisely because considering only fermionic-fermionic coupling gave qubit chains scaling in length linearly with N. In spite of these technical limitations, we find that we are able to gather interesting qualitative information about the effect of deformations to the black hole geometry on information scrambling.

Chapter 4

Results and Analysis

In this section we present the results of our simulations of information scrambling in a static deformed $N = 4$ Matrix theory black hole. Qualitative features of our results are discussed as well as possible interpretations.

4.1 Simulation Parameters

We ran each simulation with $N = 4$, corresponding to 16 qubits. Each plot shown gives the (normalized) entanglement entropy over time, calculated from the reduced density matrix describing 9 of the 16 qubits. The same 9 qubits were chosen for each simulation. We ran the simulation, with varying deformation parameters, for three different initial states. The results for each initial state were similar, so we include here the plots for only initial state "29876." The number 29876, converted into binary, corresponds to a qubit chain configuration where a "0" represents the qubit being spin down and a "1" represents the qubit being spin up.

For each simulation, we set the fuzzy sphere radius parameter $\nu = 1$. Since the Y_0^0 matrix corresponds to merely a translation, we instead deformed the sphere in the Y_0^1, Y_0^2, and Y_0^3 modes. In other words, we ran simulations with $x_3 = cY_0^j$ for $j = 1, 2, 3$, where c is a scaling constant. We also ran simulations with $x_3 = 0$, corresponding to no deformation. We ran these simulations at $c = 5$, $c = 15$, and $c = 50$. We ran the $c = 5$ and $c = 15$ simulations for 10,000 timesteps and the $c = 50$ simulations for 20,000 timesteps of half the length. In both cases, the timestep was small enough to preserve the unitary nature of our evolution. For unitary evolution, the trace of the density matrix must remain at 1. Our timesteps were small enough to keep a trace of 1, to 5 decimal points, meaning that we kept our numerical error small.

In addition to deforming the sphere in purely the Y_0^j modes, we also performed simulations in which we separated one of the D0-branes from the others by a large distance. It is possible that this could be a good model for Hawking radiation. The simulation parameters for this case (e.g. timestep, initial state) were the same as for the other cases.

In all cases, we find that the entanglement entropy approaches some equilibrium value, around which it subsequently fluctuates. We regard the first time the entropy hits its equilibrium value as the system's scrambling time. We find that the information scrambling process is extremely insensitive to deformations in the Y_0^1 mode, quite sensitive to Y_0^2 deformations, and very sensitive to Y_0^3 deformations. For large deformations, we witness strongly periodic behavior, strange for such a thermal system.

4.2 Analysis

Our results are indeed surprising. In [12], all plots were qualitatively similar. The extreme insensitivity of information scrambling to a deformation in the Y_0^1 mode, in light of the tangible effects of the Y_0^2 and Y_0^3 deformations, is strange.

Moreover, the strong periodicity of large Y_0^2 and Y_0^3 deformations show that extremely warped Matrix theory black holes have interesting information scrambling dynamics. Looking at our plots, we immediately see that as the deformation parameter c increases (i.e. the fuzzy sphere is stretched more and more), the average entropy of the system decreases. This

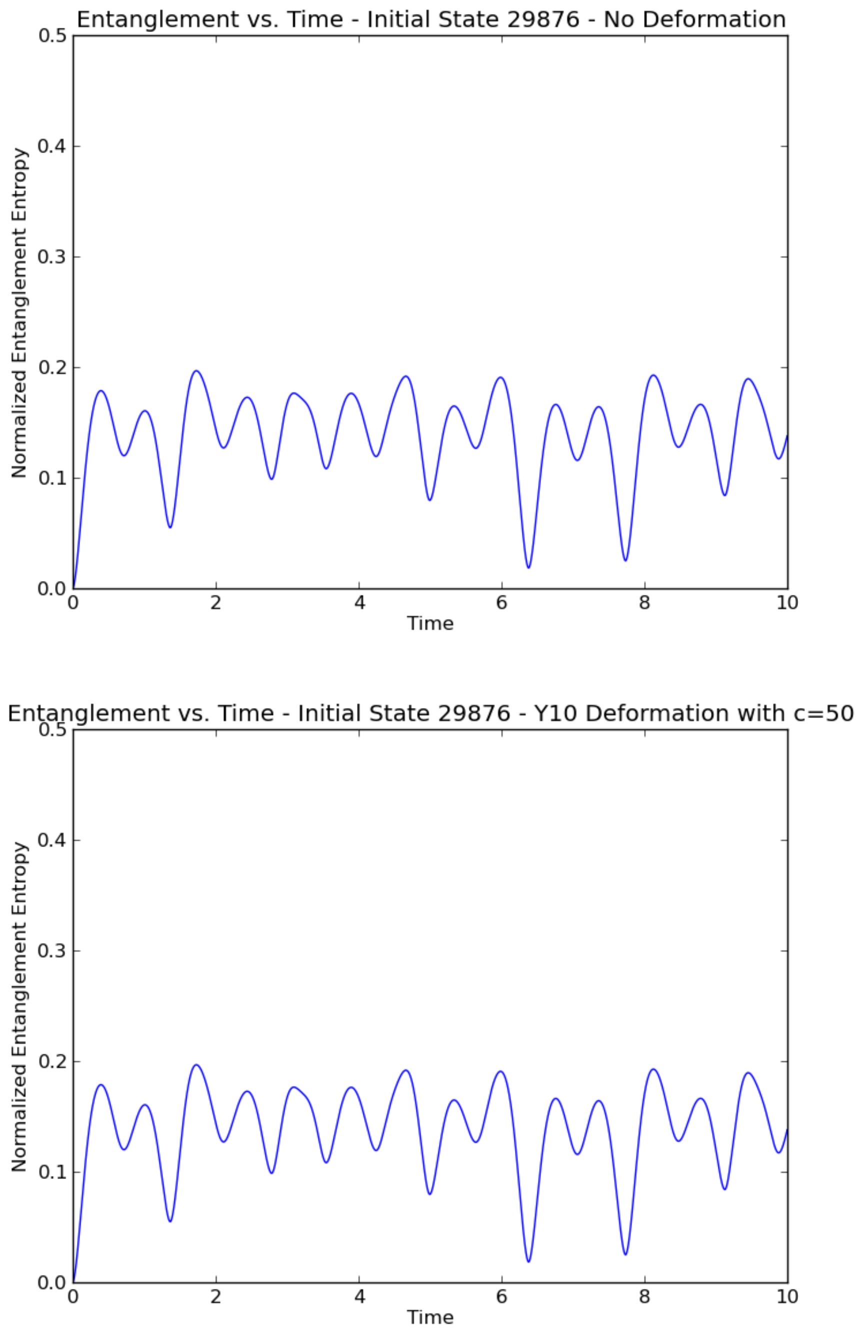


Figure 4.1: Entropy curves for no deformation and a Y_0^1 deformation with $c = 50$. Strangely, we see that an extreme deformation by Y_0^1 does not affect the information scrambling process.

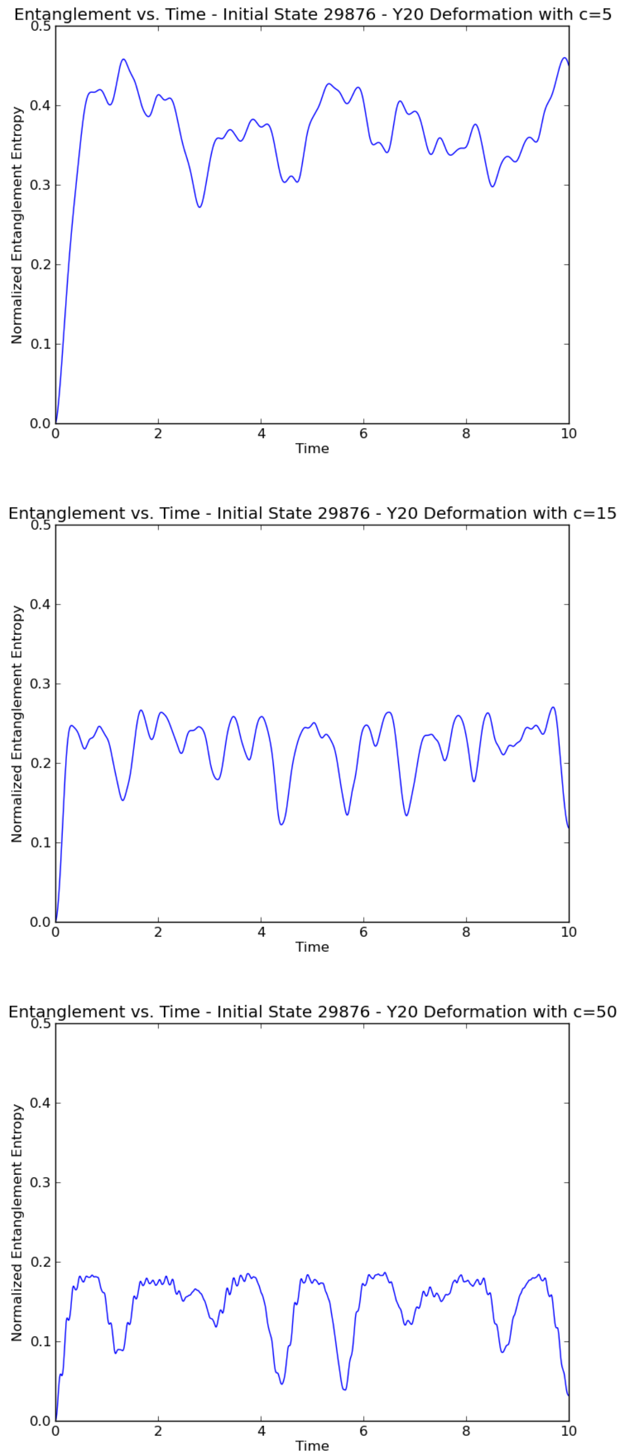


Figure 4.2: Entropy curves for Y_0^2. As c is increased, the average entropy of the thermalized system decreases. As the deformation becomes more extreme, a second timescale describing the system's pseudoperiodic behavior emerges.

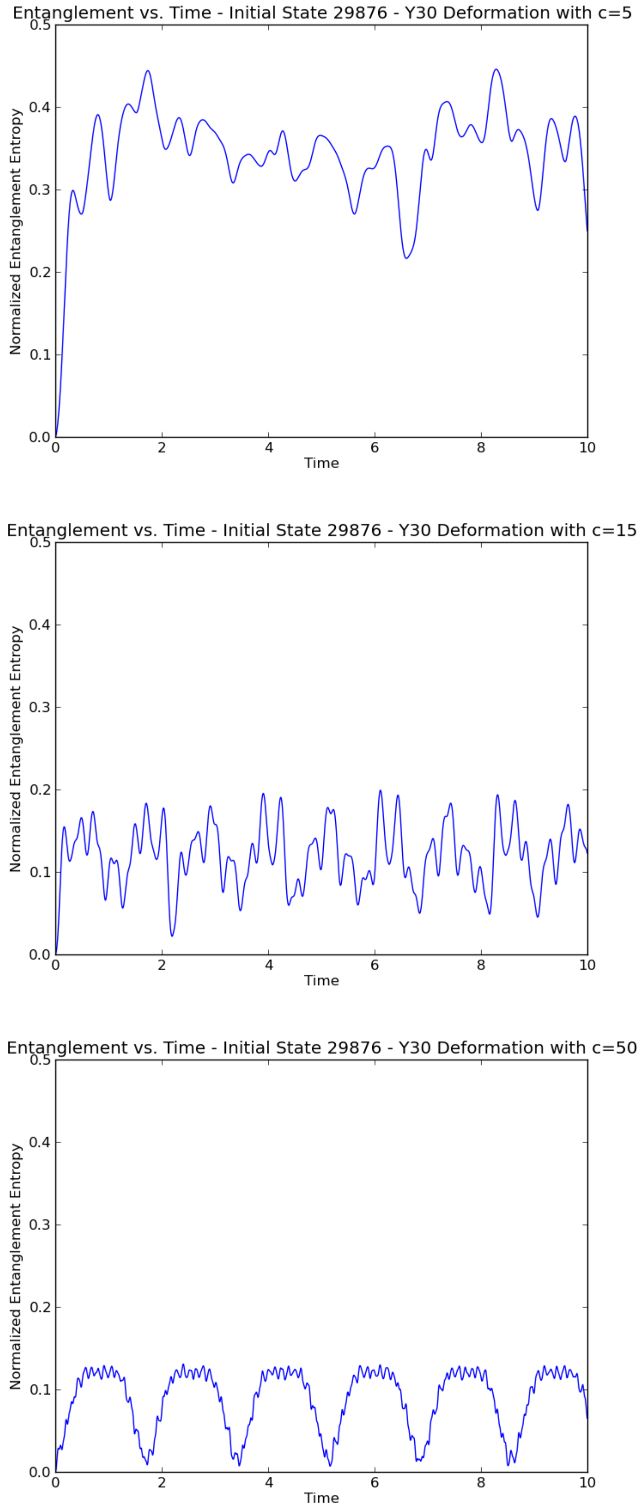
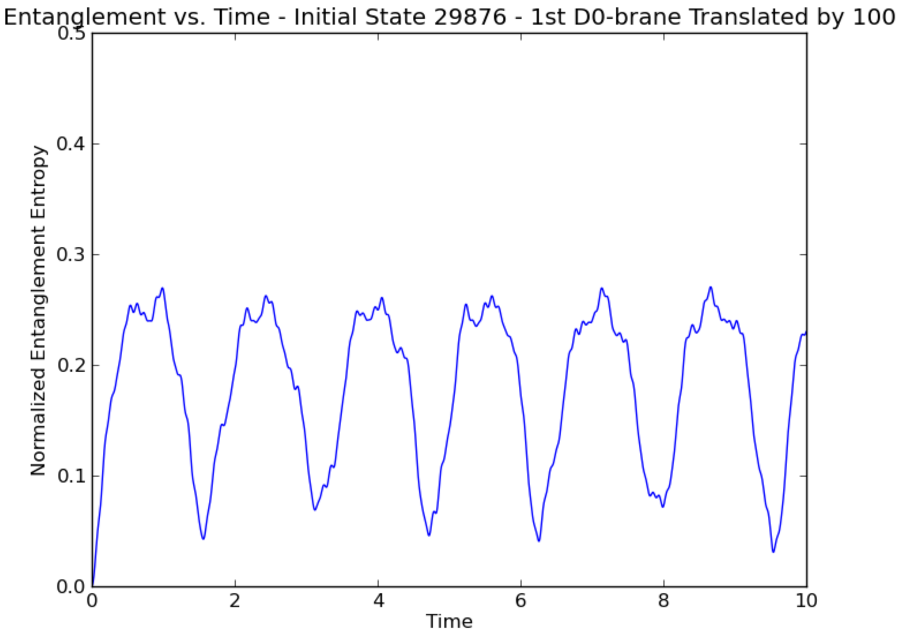


Figure 4.3: Entropy curves for Y_0^3. It seems that the scrambling process is most sensitive to this deformation mode. For $c = 50$, the entropy is strongly periodic.



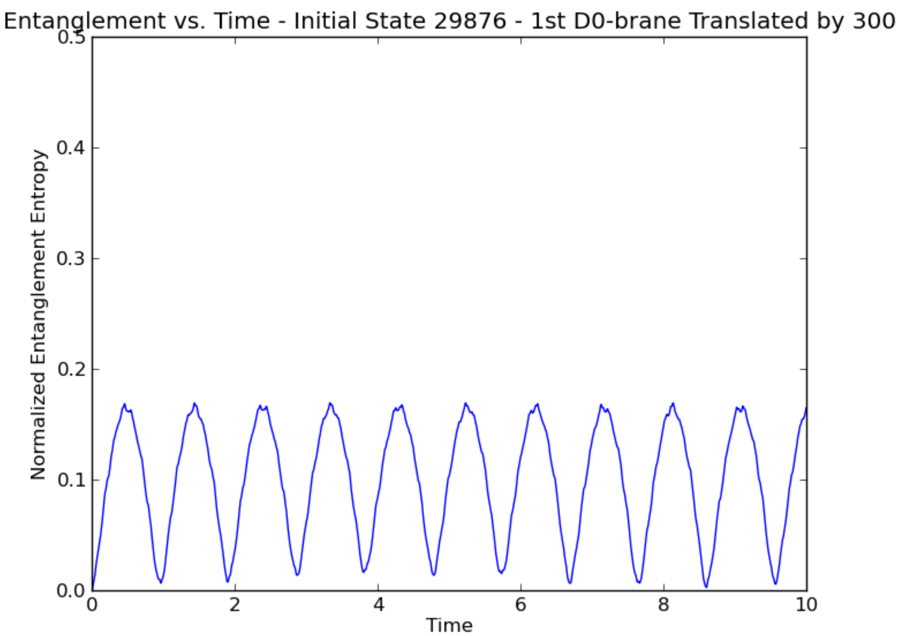


Figure 4.4: Simulations in which one of the D0-branes was sent far away from the others. This could provide some model for Hawking radiation. We see the same strong periodicity in the entropy as for the case of large Y_0^2 and Y_0^3 deformations.

suggests an interpretation that associates entanglement among qubits on the black hole with spatial proximity. Indeed, we see that for large deformations, the minimum of the entropy after thermalization scales approximately as $S_{\min} \propto \frac{1}{c}$.

It is interesting that, as c increases and thus the fermionic-bosonic coupling begins to dominate the fermionic-fermionic coupling, we see strongly periodic behavior. Moreover, for the undeformed fuzzy sphere, this strong period is not present. This suggests that the asymmetry of the bosonic deformations gives rise to periodic behavior. This idea is further supported by the fact that a deformation by Y_0^1, which stretches the sphere uniformly, doesn't affect information scrambling. A possible interpretation of the periodic nature of information scrambling on a strongly-deformed sphere is that the periods are dictated by the movement of qubits between nodes in our asymmetric D0-brane system. It is interesting to note, however, that the periodicity does not seem to scale at the same rate as the deformation parameter c. This rules out the idea that the qubits are traveling around the deformed sphere at some uniform speed. We can perhaps imagine instead that the qubits have some non-trivial spatial interaction with each other, causing complex trajectories, an N^2-body system of sorts in which the characteristic timescale of two qubits coming together and moving apart gives us the system's periodicity. There are certainly interesting, perhaps even chaotic, dynamics at play.

Chapter 5

Conclusions and Outlook

Recent years have seen much work on the information paradox. Physicists have moved beyond wondering whether infalling information obeys unitary time evolution and now consider the specifics of the unitary evolution. The goal is to understand how infalling information interacts with the black hole degrees of freedom. It is thought that the information is scrambled and emitted with Hawking radiation, avoiding information loss. Black holes may be more interesting than other information scramblers because they invoke quantum gravity and non-commutative geometry. Perhaps, as has been conjectured, these novel features endow the black hole with some ultra-efficient information scrambling mechanism.

In this thesis, we have explored the effects of (non-commutative) geometry on information scrambling in a theory of quantum gravity, analyzing the interactions between the fermionic and bosonic degrees of freedom in Matrix theory. We saw that the fermionic degrees of freedom formed a network of interacting qubits living on the black hole's event horizon, represented by our fermionic degrees of freedom. We then studied the dynamics of information scrambling among these qubits for different geometries corresponding to deformed black holes. We found that information scrambling depends on geometry in strange ways with no immediately obvious interpretation. In some sense, entanglement entropy scaled inversely with distance, but in a hard to quantify way. More simulations, perhaps involving even larger deformation parameters, are needed to quantify our results.

Moreover, our inability to test the fast scrambling conjecture might be circumvented by novel computational techniques or resorting to supercomputers to get the job done. It would also be interesting to allow dynamics in the bosonic degrees of freedom, although this would require significant modifications to our algorithm. Having dynamic bosonic degrees of freedom, however, would allow us to see what happens when one D0-brane decouples and radiates away (a potential model for Hawking radiation) in a more realistic way. There are also more tests of numerical legitimacy that we could run, including experimentation with different timesteps. In any case, there are many options to pursue, all of which promise to give us insight about the role of non-commutative geometry in information scrambling and quantum gravity.

Appendix A

A Brief Introduction to Gauge Theory

An important development in modern physics is the identification of symmetries with physical theories. In the early 1900's Emmy Noether proved that every symmetry of a physical system corresponds to some conserved quantity. One special type of symmetry is a gauge symmetry, when the physics (i.e. Lagrangian) of a system is invariant under time (and space, in the case of field theory)-dependent transformations induced by some group. In this appendix, we demonstrate the $U(N)$ gauge invariance of the Matrix theory Lagrangian.

Consider a variant of the Matrix theory Lagrangian where we write ∂_t instead of D_t:

$$L = \frac{1}{2}\mathrm{Tr}\left[(\partial_t X_i)(\partial_t X_i) + \frac{1}{2}[X_i, X_j][X_i, X_j] + \Psi\partial_t\Psi + \Psi\gamma_i[X_i, \Psi]\right]. \qquad \text{(A.1)}$$

Let N denote the dimension of our matrices. The Lagrangian (A.1) is what we call "globally invariant"[1] under unitary transformations. In other words, for some unitary matrix $U \in U(N)$, the Lagrangian (A.1) doesn't change if we make the transformations $X_i \to X_i' = UX_iU^{-1}$ and $\Psi \to \Psi' = U\Psi U^{-1}$. To see this, note that U is a constant matrix and thus

[1]In some sense, this is an abuse of language. The phrase global invariance generally refers to field theories, in which our degrees of freedom are functions on spacetime. The matrices X_i and Ψ are *not* fields: they are functions of time only. By global I refer to static unitary transformations

$\partial_t(UX_iU^{-1}) = U(\partial_t X_i)U^{-1}$, so that we have

$$(\partial_t X_i')(\partial_t X_i') = U(\partial_t X_i)U^{-1}U(\partial_t X_i) \tag{A.2}$$

$$= U(\partial_t X_i)(\partial_t X_i)U^{-1}. \tag{A.3}$$

The way our Lagrangian is written, we can see that each term inside the trace, say B, becomes UBU^{-1}. Therefore the quantity inside the trace is simply having a constant unitary transformation performed on it. Since the trace doesn't change for unitary transformations, the Lagrangian is invariant under constant unitary transformations. We call this global invariance under the *gauge group* $U(N)$.

We can give the Lagrangian an even stronger symmetry under unitary transformations. Suppose instead that U were a function of time (i.e. $U = U(t)$), and perform the same transformations. We then would have that

$$\partial_t X_i' = (\partial_t U)X_iU^{-1} + U(\partial_t X_i)U^{-1} + UX_i(\partial_t U^{-1}). \tag{A.4}$$

We can quickly see that this will screw up the invariance of our Lagrangian, since $(\partial_t X_i')(\partial_t X_i') \neq U(\partial_t X_i)(\partial_t X_i)U^{-1}$. This problem occurs for all terms involving derivatives. Looking at (A.1), we see that we can make our Lagrangian invariant under $U(t)$ if we can substitute for our conventional derivative a "covariant gauge derivative" D_t such that

$$D_t(UXU^{-1}) = U(D_tX)U^{-1}. \tag{A.5}$$

We thus define D_t by

$$D_tY = \partial_t Y - iA, \tag{A.6}$$

where A is an $N \times N$ matrix, called our *gauge* matrix. We can then ensure that (A.5) is satisfied by constraining the transformation of A. Since our Lagrangian is already globally invariant under $U(N)$, we can write $U = e^{i\epsilon G}$, where $\epsilon \ll 1$ and $G = G(t)$ is some hermitian matrix. Since ϵ is small, we can approximate U by $U = e^{i\epsilon G} \approx 1 + i\epsilon G$. Similarly, $U^{-1} \approx$

$1 - i\epsilon G$. Then we have that

$$X' = UXU^{-1} \approx (1 + i\epsilon G)X(1 - i\epsilon G) = X + i\epsilon[G, X]. \tag{A.7}$$

Let's solve for the transformation rule of A. Plugging the transformation (A.7) into our requirement $D_t(X')$, we have that

$$D_t(X') = \partial_t X - i[A', X] + i\epsilon(\partial_t GX + G\partial_t X - \partial_t XG - X\partial_t G) - i[A', i\epsilon[G, X], \tag{A.8}$$

while

$$UD_t(X)U^{-1} = \partial_t X - i[A, X] + i\epsilon[G, \partial_t X, i[A, X]]. \tag{A.9}$$

Equating these two expressions, grouping a few terms into commutators, and canceling the $\partial_t X$ term and a $i\epsilon[G, \partial_t X]$ term, we have

$$-i[A', X] + i\epsilon[\partial_t G, X] - i[A', i\epsilon[G, X]] = -i[A, X] - i\epsilon[G, -i[A, X]], \tag{A.10}$$

from which we can move the $i\epsilon[\partial_t G, X]$ to the other side, divide by $-i$, and expand and rearrange a few commutators to get

$$[A', X] + [A', i\epsilon GX] - [A' + i\epsilon XG] = [A + \epsilon\partial_t G, X] - [G, i\epsilon AX] + [G, i\epsilon XA]. \tag{A.11}$$

If we let $A' = A + i\epsilon XG$, what happens? The first terms on each side of the equation cancel out, and we're left with, expanding the commutators

$$i\epsilon[A'GX - GXA' - A'XG + XGA'] = i\epsilon[AXG - GXA - XAG + GXA] \tag{A.12}$$

$$i\epsilon[\epsilon\partial_t GXG - GX\epsilon\partial_t G - X\epsilon\partial_t GG + GX\epsilon\partial_t G] = 0 \tag{A.13}$$

$$O(\epsilon^2) = 0, \tag{A.14}$$

which is true since ϵ is taken to be very small. Thus our transformation rule for A is

$$A' = A + \epsilon\partial_t G. \tag{A.15}$$

Note that if $A = 0$, applying some constant unitary transformation to the system gives us that $A' = 0$, since $\partial_t U = 0$ implies that $\partial_t G = 0$. Taking this fact into consideration, we only need to explore diagonal matrix deformations, since we may diagonalize Hermitian matrices in our simulation without changing the physics.

Appendix B

Spinors

In this section we derive Majorana-Weyl spinors from the Clifford algebra for the Dirac matrices. Denote the dimension of spacetime by d. Denote by $\eta^{\mu\nu}$ the d-dimensional Minkowski metric with signature $(-, +, \ldots, +)$.

The Dirac equation[1] describes relativistic quantum mechanics for spin-$\frac{1}{2}$ particles. In his derivation, Dirac found that the wavefunctions in his equation had to be multiplied on the left by objects γ^μ (with $\mu = 0, \ldots, d-1$) obeying the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{B.1}$$

Since this sort of relation cannot be realized for ordinary numbers, Dirac decided to represent the γ^μ as matrices, since matrices can obey nontrivial anticommutation relations. Dirac then realized that this necessitated that the wavefunctions for spin-$\frac{1}{2}$ particles be *vectors* packaging several wavefunctions together. The vectors acted on by the γ^μ are called *spinors*. Here, following [15], we derive the conditions for Dirac, Majorana, and Weyl spinors, as well as listing the representation of the Dirac matrices we use in this thesis.

[1]For a more complete description of the Dirac equation and how it involves the Dirac matrices, see [14]

B.1 Spinors

Take the dimension of spacetime to be even, with $d = 2k+2$. Important cases are $d = 4$ (corresponding to $k = 1$) and $d = 10$ (corresponding to $k = 4$). We then group the γ^μ into $k+1$ sets of anticommuting raising and lowering operators,

$$\gamma^{0\pm} = \frac{1}{2}(\pm\gamma^0 + \gamma^1), \tag{B.2}$$

$$\gamma^{a\pm} = \frac{1}{2}(\gamma^{2a} \pm \gamma^{2a+1}), \qquad a = 1, \dots, k. \tag{B.3}$$

One may show through exhaustive calculation that

$$\{\gamma^{a+}, \gamma^{b-}\} = \delta_{ab}, \tag{B.4}$$

$$\{\gamma^{a+}, \gamma^{b+}\} = \{\gamma^{a-}, \gamma^{b-}\} = 0, \tag{B.5}$$

where $a, b = 0, \dots, k$. These anticommutation relations are shown by brute force, plugging (B.2) and (B.3) into the anticommutator and using (B.1). For example,

$$\begin{aligned}\{\gamma^{0\pm}, \gamma^{a\pm}\} =& \frac{1}{4}\left(\pm\gamma^0\gamma^{2a} + i\gamma^0\gamma^{2a+1} + \gamma^1\gamma^{2a} \pm i\gamma^1\gamma^{2a+1}\right.\\ &\left.\pm\gamma^{2a}\gamma^0 + \gamma^{2a}\gamma^1 + i\gamma^{2a+1}\gamma^0 \pm i\gamma^{2a+1}\gamma^1\right) &\text{(B.6)}\\ =& \frac{1}{4}\left(\pm\{\gamma^0\gamma^{2a}\} + i\{\gamma^{2a+1}, \gamma^0\} + \{\gamma^{2a}, \gamma^1\} \pm i\{\gamma^{2a+1}, \gamma^1\}\right) &\text{(B.7)}\\ =& 0. &\text{(B.8)}\end{aligned}$$

In particular, (B.5) gives us that $(\gamma^{a+})^2 = (\gamma^{a-})^2 = 0$ for each a.

We will now show that this gives us a spinor ζ such that

$$\gamma^{a-}\zeta = 0 \quad \text{for all } a. \tag{B.9}$$

Take v to be a nonzero spinor acted on by the $\gamma^{a\pm}$. If $\gamma^{0-}v \neq 0$, define a new spinor v_0 by $v_0 = \gamma^{0-}v$. Otherwise define v_0 by $v_0 = v$. If $\gamma^{1-}v_0 \neq 0$, let $v_1 = \gamma^{1-}v_0$. Otherwise let $v_1 = v_0$. Performing this process through k iterations gives us a spinor v_k, which will be our ζ. Why is this? We have that

$$\zeta = \gamma^{a_1-}\gamma^{a_2-}\dots\gamma^{a_n-}v, \tag{B.10}$$

where $a_1, \ldots, a_n$ are the a's for which $\gamma^{a-} v_a \neq 0$. Now consider the action of γ^{a-}, where $a = a_m$, on ζ. Then we have

$$\begin{aligned} \gamma^{a_m -}\zeta &= (-1)^m \gamma^{a_1 -} \ldots \gamma^{a_m -}\gamma^{a_m -} \ldots \gamma^{a_n -} v \\ &= 0, \end{aligned} \tag{B.11}$$

where we have used the fact that $\gamma^{a_m -}\gamma^{a_m -} = 0$ and $\gamma^a \gamma^b = -\gamma^b \gamma^a$. Now consider the case where $a \neq a_m$ for all m. Then we have

$$\begin{aligned} \gamma^{a-}\zeta &= (-1)^p C \gamma^{a-} v_a \\ &= 0, \end{aligned} \tag{B.12}$$

where C is the product of p of the γ^2 and we have used the fact that $a \neq a_m$ for all m implies that $\gamma^{a-} v_a = 0$. Therefore we have that ζ obeys (B.9).

Given ζ, we may obtain a spinor representation of dimension 2^{k+1} ($= 32$ for $d = 10$ and 4 for $d = 4$) by acting on ζ in all possible ways with the γ^{a+}, at most once for each a value since $(\gamma^{a+})^2 = 0$. We label our basis states by $\vec{s} = (s_0, \ldots, s_k)$, where each s_a is $\pm\frac{1}{2}$:

$$\zeta^{\vec{s}} \equiv (\gamma^{k+})^{s_k + \frac{1}{2}} \ldots (\gamma^{0+})^{s_0 + \frac{1}{2}} \zeta. \tag{B.13}$$

Our original ζ then corresponds to $\vec{s} = (-\frac{1}{2}, \ldots, -\frac{1}{2})$; from ζ we generate our entire basis by raising various s_a from $-\frac{1}{2}$ to $+\frac{1}{2}$ using γ^{a+r}. Taking the $\zeta^{\vec{s}}$ to be our basis states, it is straightforward to derive the matrix elements of the γ^μ via the definitions and anticommutation relations. Our matrices are $2^{d/2} \times 2^{d/2}$. Thus increasing d by two doubles the size of our Dirac matrices, so we can conceivably iterate from lower-dimensional matrix representations using the Kronecker product. For $d = 2$, we have that

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{B.14}$$

[2] p is the number of $\gamma^{a_m -}$ that γ^{a-} must move through to act on v_a.

It is easy to show that, given a representation of the Dirac matrices in $d = 2k+2$, the matrices

$$\gamma^\mu = \gamma_k^\mu \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mu = 0, \ldots, d-3, \tag{B.15}$$

$$\gamma^{d-2} = I_k \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^{d-1} = I_k \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{B.16}$$

with γ_k^μ the $2^k \times 2^k$ Dirac matrices in $d-2$ dimensions and I_k the $2^k \times 2^k$ identity matrix, satisfy the correct anticommutation relations. We are taking our $2^k \times 2^k$ representation and using the Kronecker product with the Pauli matrices to get a $2^{k+1} \times 2^{k+1}$ representation.

The γ^μ can be used to define the Lorentz algebra (i.e. the algebra of Lorentz transformations). The generators of the Lorentz algebra, written $\Sigma^{\mu\nu}$ and defined by

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{B.17}$$

satisfy the Lorentz algebra.

$$i[\Sigma^{\mu\nu}, \Sigma^{\sigma\rho}] = \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\sigma}\Sigma^{\nu\rho}. \tag{B.18}$$

It can be shown that the Lorentz generators $\Sigma^{2a,2a+1}$ commute with each other. Thus they can be simultaneously diagonalized. For each a, we define

$$S_a \equiv i^{\delta_{a,0}}\Sigma^{2a,2a+1} = \gamma^{a+}\gamma^{a-} - \frac{1}{2}. \tag{B.19}$$

By definition, each $\zeta^{\vec{s}}$ is a simulataneous eigenstate of each S_a with eigenvalues s_a. Using these spinors as a basis for the Lorentz algebra, we have what we call the 2^{k+1}-dimensional *Dirac* representation of the Lorentz algebra. Because each $\Sigma^{\mu\nu}$ is quadratic in the γ matrices and each γ matrix can decomposed into a linear combination of raising and lowering operators, we have each $\Sigma^{\mu\nu}$ can only change either two or none of the s_a of some $\zeta^{\vec{s}}$. In other words, the $\zeta^{\vec{s}}$ with even and odd numbers of $+\frac{1}{2}$s do not mix under the Lorentz generators. Thus we say that the Dirac representation of the Lorentz algebra is *reducible*: we can decompose it into multiple irreducible representations of the Lorentz algebra. To see this, define

$$\gamma = i^{-k}\gamma^0\gamma^1 \dots \gamma^{d-1}, \tag{B.20}$$

which has the properties

$$(\gamma)^2 = 1, \quad \{\gamma, \gamma^\mu\} = 0, \quad [\gamma, \Sigma^{\mu\nu}] = 0. \tag{B.21}$$

Note that

$$\gamma = 2^{k+1} S_0 S_1 \dots S_k, \tag{B.22}$$

so that γ, written in the basis of $\zeta^{\vec{s}}$, is diagonal, taking the value $+1$ when the s_a include an even number of $-\frac{1}{2}$s and -1 when the s_a include an odd number of $-\frac{1}{2}$s. The 2^k states with a γ eigenvalue (called *chirality*) of $+1$ form what we call a *Weyl* representation of the Lorentz algebra, while the 2^k states with eigenvalue -1 form a second, distinct Weyl representation. In other words, we may decompose the 2^{k+1}-dimensional Dirac representation of spinors into two Weyl representations, each of dimension 2^k. Note that this corresponds to a 2-dimensional representation for $d = 4$ and a 16-dimensional representation for $d = 10$.

B.1.1 Majorana spinors

Our construction of the γ matrices for even spacetime dimension $d = 2k + 2$ is unique up to change in basis. Thus, since

$$\begin{aligned} \{\gamma^{\mu *}, \gamma^{\nu *}\} &= \gamma^{\mu *}\gamma^{\nu *} + \gamma^{\nu *}\gamma^{\mu *} \\ &= \{\gamma^\mu, \gamma^\nu\}^* && \text{(B.23)} \\ &= (2\eta^{\mu\nu})^* = 2\eta^{\mu\nu}, && \text{(B.24)} \end{aligned}$$

the matrices $\gamma^{\mu *}$ (and $-\gamma^{\mu *}$) satisfy the same Clifford algebra as the γ^μ matrices, and thus must be related by a similarity transformation. By construction, in our $\zeta^{\vec{s}}$, the matrix elements of $\gamma^{a\pm}$ are real, so we have from definitions (B.2) and (B.3) that $\gamma^3, \gamma^5, \ldots, \gamma^{d-1}$ are imaginary and the other γ^μ are real. Define

$$B_1 = \gamma^3\gamma^5 \ldots \gamma^{d-1}, \quad B_2 = \gamma B_1. \tag{B.25}$$

Then we have that

$$B_1 \gamma^\mu B_1^{-1} = \gamma^3\gamma^5 \ldots \gamma^{d-1}\gamma^\mu(\gamma^{d-1})^{-1} \ldots (\gamma^3)^{-1}. \tag{B.26}$$

We next move the γ^μ to the left side of the right side of the equation. If $\mu \neq 3, 5, \ldots, d-1$, we pick up a factor of (-1) for each place γ^μ must move, for a total of $(-1)^{\frac{d}{2}-1} = (-1)^k$. Then we have

$$B_1 \gamma^\mu B_1^{-1} = (-1)^k \gamma^\mu B_1 B_1^{-1} = (-1)^k \gamma^\mu = (-1)^k \gamma^{\mu *}, \tag{B.27}$$

where we have used the fact that γ^μ is real in this case. If $\mu = 3, 4, \ldots, d-1$, we pick up one fewer factor of (-1), since $\gamma^\mu \gamma^\mu = \nu^{\mu\mu} = 1$ in this case. Then we have

$$B_1 \gamma^\mu B_1^{-1} = (-1)^{k-1} \gamma^\mu B_1 B_1^{-1} = (-1)^k(-\gamma^\mu) = (-1)^k \gamma^{\mu *}, \tag{B.28}$$

where we have used the fact that γ^μ is complex in this case. Thus we have that, in general

$$B_1 \gamma^\mu B_1^{-1} = (-1)^k \gamma^{\mu *}. \tag{B.29}$$

By similar means, one may show that

$$B_2 \gamma^\mu B_2^{-1} = (-1)^{k+1} \gamma^{\mu *}. \tag{B.30}$$

Using these facts, one may show that, for $B = B_1$ or $B = B_2$,

$$B \Sigma^{\mu\nu} B^{-1} = -\Sigma^{\mu\nu *}. \tag{B.31}$$

It is then possible to show that the spinors ζ and $B^{-1}\zeta^*$ transform identically under the Lorentz group, so the Dirac representation is its own conjugate. Moreover, acting on the chirality matrix γ, one may show that

$$B_1 \gamma B_1^{-1} = B_2 \gamma B_2^{-1} = (-1)^k \gamma^*. \tag{B.32}$$

Thus transforming by B will change the eigenvalue of γ when k is odd and not when it is even. Moreover, for even k each Weyl representation is its own conjugate, while for odd k each Weyl representation is conjugate to the other.

We may also enforce on spinors a *Majorana* condition, which relates ζ^* to ζ. Ensuring that this condition be consistent with Lorentz transformations give us the form

$$\zeta^* = B\zeta, \tag{B.33}$$

with B satisfying (B.31). Taking the conjugate of this expression gives us that $\zeta = B^*\zeta^* = B^*B\zeta$. In other words, our condition is consistent if and only if $B^*B = 1$. Then, using the reality and anticommutation properties of our γ matrices, one may show that

$$B_1^* B_1 = (-1)^{k(k+1)/2}, \quad B_2^* B_2 = (-1)^{k(k-1)/2}. \tag{B.34}$$

Thus a Majorana condition using B_1 is possible only if $k = 0 \pmod 4$ or $3 \pmod 4$, and using B_2 only if $k = 0 \pmod 4$ or $1 \pmod 4$. Moreover, for $k = 0 \pmod 4$ both conditions are possible but physically equivalent, since B_1 and B_2 are related by a similarity transformation.

Finally, we may impose a Majorana condition on a Weyl spinor only if $B^*B = 1$ *and* the Weyl representation is conjugate to itself, since a representation must be closed under conjugation for (B.33) to make sense. Then, since for odd k each Weyl representation is *not* self-conjugate, we cannot impose both the Majorana and Weyl conditions on a spinor; we can only impose one or the other. For $k = 0 \bmod 4$ (i.e. $d = 2 \bmod 8$), however, a spinor can simultaneously satisfy the Majorana and Weyl conditions. Majorana-Weyl spinors in $d = 10$ play an important role in string theory. It is these spinors that appear in the Lagrangian for Matrix theory.

B.2 Our γ-Matrix Representation

To make the qubit-qubit interaction in our Matrix theory Hamiltonian simpler, we choose a specfic representation of the γ matrices. Of course, the representation we choose has no effect on the physics at hand. The network of interactions is the same for any choice of representation, but our choice affects how the expanded Hamiltonian looks. We use the same representation as [12]:

$$\gamma_1 = \begin{pmatrix} 0_4 & 0_4 & 0_4 & M \\ 0_4 & 0_4 & -M & 0_4 \\ 0_4 & -M & 0_4 & 0_4 \\ M & 0_4 & 0_4 & 0_4 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0_4 & 0_4 & 1_4 & 0_4 \\ 0_4 & 0_4 & 0_4 & 1_4 \\ 1_4 & 0_4 & 0_4 & 0_4 \\ 0_4 & 1_4 & 0_4 & 0_4 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} -1_4 & 0_4 & 0_4 & 0_4 \\ 0_4 & -1_4 & 0_4 & 0_4 \\ 0_4 & 0_4 & 1_4 & 0_4 \\ 0_4 & 0_4 & 0_4 & 1_4 \end{pmatrix}, \tag{B.35}$$

where 1_4 is the 4×4 identity matrix, 0_4 is the 4×4 zero matrix, and M is defined by

$$M \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.36}$$

Bibliography

[1] John Preskill. Quantum computation lecture notes, 1998.

[2] Leonard Susskind, Lárus Thorlacius, and John Uglum. The stretched horizon and black hole complementarity. *Phys. Rev. D*, 48:3743–3761, Oct 1993. http://link.aps.org/doi/10.1103/PhysRevD.48.3743.

[3] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, and James Sully. Black holes: Complementarity or firewalls? 2012. http://arxiv.org/abs/1207.3123.

[4] Patrick Hayden and John Preskill. Black holes as mirrors: quantum information in random subsystems. *Journal of High Energy Physics*, 2007(09):120, 2007.

[5] Yasuhiro Sekino and Leonard Susskind. Fast scramblers. 2008. http://arxiv.org/abs/0808.2096.

[6] Leonard Susskind. Addendum to fast scramblers. *arXiv preprint arXiv:1101.6048*, 2011.

[7] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind. M theory as a matrix model: A conjecture, 1996. http://arxiv.org/abs/hep-th/9610043.

[8] Keshav Dasgupta, Mohammad M. Sheikh-Jabbari, and Mark Van Raamsdonk. Matrix Perturbation Theory For M-theory On a PP-Wave. *Journal of High Energy Physics*, 2002(05):056–056, May 2002. arXiv: hep-th/0205185.

[9] Tom Banks, W. Fischler, Igor R. Klebanov, and Leonard Susskind. Schwarzschild black holes from matrix theory. *Physical Review Letters*, 80(2):226, 1998.

[10] Tom Banks, Willy Fischler, Igor R. Klebanov, and Leonard Susskind. Schwarzschild black holes in Matrix theory II. *Journal of High Energy Physics*, 1998(01):008, 1998.

[11] Gary T. Horowitz and Emil J. Martinec. Comments on black holes in matrix theory. *Physical Review D*, 57(8):4935, 1998.

[12] Lucas Brady and Vatche Sahakian. Scrambling with matrix black holes. *Phys. Rev. D*, 88:046003, Aug 2013. `http://link.aps.org/doi/10.1103/PhysRevD.88.046003`.

[13] Robert L. Borrelli and Courtney S. Coleman. *Differential Equations: A Modeling Perspective*. Wiley, January 2004.

[14] David Griffiths. *Introduction to Elementary Particles*. Wiley-VCH, 2008.

[15] Joseph Polchinski. *String Theory, Vol. 2*. Cambridge University Press, 2005.